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SIS model on homogeneous networks with threshold type delayed contact reduction

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ABSTRACT

We study the dynamics of a delayed SIS epidemic model on homogeneous networks, where it is assumed that individuals modify their contact patterns upon realizing the risk of infection. This decision is made with some time delay, and it is threshold type: when the density of infected nodes reaches a critical value, the number of links is reduced by a given factor. Such assumptions lead to a delay differential equation with discontinuous right hand side. We show that if the basic reproduction number $R_0 \leq 1$, then the disease will be eradicated, while it persists for $R_0 > 1$. In the latter case, there is a globally asymptotically stable endemic equilibrium, except for a crucial interval of reproduction numbers, where the system shows oscillations. We construct explicitly the unique slowly oscillatory periodic solution, which has strong attractivity properties, and show the existence of rapidly oscillatory periodic solutions with any frequency. The amplitude of the oscillations is determined by the time delay. Our results indicate that with such information delays, the link density of a network has an important effect on the qualitative dynamics of infectious diseases.

1. Introduction

In recent years, many researchers have studied the spread of infections on networks [1–10], focusing mostly on two different complex networks: the Watts–Strogatz model (a relatively homogeneous network exhibiting small-world properties [11]) and the Barabási–Albert model (a typical example of a scale-free network [12]). Usually they have used SIS type models as a framework, where the infection spreads via connections between S and I type nodes of the network. However, there are interesting phenomena that has not been explored yet, even for homogeneous networks.

In many disease transmission models, time delay plays an important role in several epidemiological mechanisms. Epidemic models with time delays have been extensively studied in various contexts (see, for example [13–16]). Recently there have been some studies about epidemic models on complex networks with time delays. For example, in the paper [17], the authors present a modified SIS model with the effect of time delay in the transmission on small-world and scale-free networks. They found that the presence of the delay may enhance outbreaks and increase the prevalence of infectious diseases in these networks. Another recent work is [18], where the authors considered a delayed SIR epidemic model on an uncorrelated complex network and addressed the effect of time lag on the shape and the number of epidemic waves. They showed that a large delay can cause multiple waves with larger amplitudes in the second and subsequent waves.

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Depending on the perception of the risk of infections, individuals may reduce their usual activities after receiving information about the epidemic outbreak. For example, people may reduce the time that they go out, they avoid crowded places or mass transportation, school closures can be applied, hospital visitations can be restricted, and so on. Since information on the ongoing epidemics motivates people to change their behaviors, it may impact the disease dynamics itself [19], as the contact network changes [20]. There are many factors contributing to the fact that the exact number of infected individuals cannot be known in real time, such as the incubation period of diseases, the concealment of infected individuals, or the time needed for collecting and analyzing epidemic data. Based on the above facts, in this paper we assume that individuals adjust their connections according to some information delay concerning the actual disease prevalence. For simplicity, we work with the assumption that individuals uniformly and randomly reduce the number of their connections by some factor whenever the density of infected individuals reaches a given threshold. We construct a model that takes into account the time delay in making this decision, and then we study the dynamics of the spread of the disease under such circumstances.

By theoretical analysis we conclude that depending on the structure of network and some key parameters, the disease will be eradicated, or the density converges to an endemic state, or oscillates in a periodic pattern. The oscillatory behavior is dominated by a slowly oscillatory solution with strong attractivity properties, but rapidly oscillatory solutions also occur with any frequency. All these behaviors qualitatively appear for any value of the delay, and they are completely characterized by the basic reproduction number which is proportional to the density of links in the network.

The paper is organized as follows. In the following section, we construct an SIS model on homogeneous networks with delayed reduction of contacts, which can be considered as a delayed relay system. In Section 3, we analyze the existence and stability of the equilibria. Section 4 includes the main results regarding slow and rapid oscillations. Some numerical simulations illustrating the key points of the theoretical analysis are given in Section 5, and we finish the paper with the conclusions.

2. The model description

We consider a susceptible–infected–susceptible (SIS) model on homogeneous networks. In the SIS model, infectious (I) individuals contaminate their susceptible (S) neighbors with some transmission rate. Meanwhile, infected individuals recover at some rate and return to the susceptible state again. By using the mean-field approach on homogeneous networks, the authors arrived to the following epidemic model in [9]:

$$\frac{dI(t)}{dt} = -\mu I(t) + \beta\langle k\rangle I(t)(1 - I(t)). \tag{1}$$

Here $I(t) \in [0, 1]$ denotes the density of infected nodes at time t. The first term considers infected nodes recovering with rate μ. The second term on the right-hand side of Eq. (1) represents the newly infected nodes. This is proportional to the transmission rate β, the number of links emanating from each node $\langle k\rangle$, and the probability that a given link points to a healthy node, which is $1 - I(t)$. Here μ, β, $\langle k\rangle$ are positive constants.

We suppose that individuals modify their links according to the information they learn on the disease spread. If the disease is not widespread ($I(t)$ is small), people remain in contact with others as usual. With the increasing number of the infected individuals, people reduce their activities and temporarily terminate some of their links if the density of infectious nodes reaches a threshold quantity. We assume this is governed by the following function:

$$h(I) = \begin{cases} 1, & I < p, \\ q, & I \geq p, \end{cases}$$

where $0 < p, q < 1$. When $I < p$ the number of links of individuals is the same as usual $\langle k\rangle$; when $I \geq p$, the links of individuals are reduced to a lower level $q\langle k\rangle$. By assuming a time delay $\tau > 0$ in making this reduction, we obtain the following epidemic model with discontinuous right hand side:

$$\frac{dI(t)}{dt} = -\mu I(t) + \beta\langle k\rangle h(I(t - \tau))I(t)(1 - I(t)). \tag{2}$$

For the sake of simplicity, we rescale the time by $\tilde{I}(t) = I(\mu^{-1}t)$. Let $\tilde{\tau} = \tau\mu$, and write the equation for $\tilde{I}(t)$:

$$\frac{d\tilde{I}(t)}{dt} = -\tilde{I}(t) + \frac{\beta\langle k\rangle}{\mu} h(I(t - \tilde{\tau}))\tilde{I}(t)(1 - \tilde{I}(t)). \tag{3}$$

Dropping the tilde to use the notation $I(t)$ for our variable in the rescaled time, and using the notation $R_0 = \frac{\beta\langle k\rangle}{\mu}$, Eq. (2) is transformed into the scalar delay differential equation

$$\frac{dI(t)}{dt} = -I(t) + R_0 h(I(t - \tau))I(t)(1 - I(t)), \tag{4}$$

where R_0 is the basic reproduction number, which expresses the number of secondary infections generated by a single infected node in a fully susceptible homogeneous network. As it is well known, scalar delay differential equations may exhibit complicated behavior if the nonlinearity is nonmonotone [21].

A dynamical system is called a delayed relay system (see Sieber [22]), if it is governed by a differential equation of the form

$$\frac{dI(t)}{dt} = \begin{cases} f_1(I(t)), & \text{if } g(I(t-\tau)) < 0, \\ f_2(I(t)), & \text{if } g(I(t-\tau)) \geq 0, \end{cases}$$

where $\tau > 0$, and f_1, f_2 are Lipschitz continuous. The switching function g is a piecewise smooth Lipschitz continuous function with the property that $g'(x) \neq 0$ whenever $g'(x)$ exists and $g(x) = 0$. The set $\{I : g(I) = 0\}$ is called the switching manifold.

Eq. (4) induces a delayed relay system with

$$f_1(I) = -I + R_0 I(1-I), \qquad f_2(I) = -I + qR_0 I(1-I)$$

and switching function $g(I) = I - p$. In our case the singleton set $\{p\}$ is the switching manifold. Let Φ_j be the flow corresponding to the ordinary differential equation $I'(t) = f_j(I(t)), j \in \{1, 2\}$. It is a straightforward calculation to show that if $R_0 > 1$, then

$$\Phi_1(t, I) = \frac{I(R_0 - 1)}{IR_0 + [(1-I)R_0 - 1]e^{-(R_0-1)t}} \tag{5}$$

and

$$\Phi_2(t, I) = \begin{cases} \dfrac{I(qR_0 - 1)}{IqR_0 + [(1-I)qR_0 - 1]e^{-(qR_0-1)t}}, & \text{if } qR_0 \neq 1, \\ \dfrac{I}{It+1}, & \text{otherwise.} \end{cases} \tag{6}$$

From our examinations we exclude the special cases $R_0 = \frac{1}{1-p}$ and $R_0 = \frac{1}{q(1-p)}$ because then one of the flows is not transversal to the switching manifold.

We have the following simple observation: if $I(t)$ is a continuous solution of Eq. (4), $T_2 > T_1$ and $I(t-\tau) < p$ for all $t \in (T_1, T_2)$, then we have $I(t) = \Phi_1(t - T_1, I(T_1))$ for all $t \in [T_1, T_2]$. Analogously, if $I(t-\tau) \geq p$ for all t in some interval, then $I(t) = \Phi_2(t - T_1, I(T_1))$ for all t in the interval.

The natural phase space for our system is $C = C([-\tau, 0], \mathbb{R})$, the Banach space of real continuous functions defined on $[-\tau, 0]$ with the usual supremum norm. If the history function $\phi \in C$ has the property that $\phi(\theta) = p$ only for finitely many $\theta \in [-\tau, 0]$, let us say, $\theta_1, \theta_2, \ldots, \theta_k$, then the forward evolution of the solution on $[0, \tau]$ will follow one of the flows on each interval $(\theta_i + \tau, \theta_{i+1} + \tau)$; thus the solution is determined by the switching times θ_j and $\phi(0)$. This way, by the method of steps, there exists a unique solution. There are many elements of C for which $\phi(\theta) - p$ has infinitely many zeros. Then we can define solutions as functions satisfying the variation of constants formula

$$I(t) = I(0) + \int_0^t I(s)[R_0 h(I(s-\tau))(1 - I(s)) - 1]ds$$

for $t \geq 0$. To avoid unnecessary technicalities, in this paper we consider only initial functions with finitely many switching times. Then we have the usual properties of existence, uniqueness and continuous dependence on initial data.

Throughout the paper by $I'(t)$ we mean the right derivative when $I(t-\tau) = p$; this will not cause any confusion.

Given the interpretation of the model, we only consider solutions $I(t) \in [0, 1]$. As solutions of (4) satisfy

$$I(t) = I(t_0) \exp\left\{\int_{t_0}^t [R_0 h(I(s-\tau))(1 - I(s)) - 1]ds\right\}, \tag{7}$$

it follows that nonnegative solutions remain nonnegative for all future time. Also note that if $I(t) = 1$, then $I'(t) = -1$. Hence solutions from the interval $[0, 1]$ remain in $[0, 1]$ for all future time of their existence. This invariance of $[0, 1]$ implies that such solutions exist for all future times. We consider the following phase space:

$$X := \{\varphi \in C : 0 \leq \varphi(\theta) \leq 1 \text{ for all } \theta \in [-\tau, 0] \text{ and } \#\{\theta : \varphi(\theta) = p\} < \infty\}.$$

Notice that if $R_0 < \frac{1}{1-p}$, then $I(t) = p$ implies $I'(t) < 0$, if $R_0 > \frac{1}{q(1-p)}$, then $I(t) = p$ implies $I'(t) > 0$, while for $\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}$ we show later in Lemma 4.2 that the number of switching times on a time interval of length τ cannot increase; thus the set X is indeed invariant.

For a solution $I : [-\tau, \infty) \to \mathbb{R}$ of Eq. (4) and for $t \geq 0$, the segment I_t is the element of X with $I_t(s) = I(t+s)$, $s \in [-\tau, 0]$. A function $I^\varphi : [-\tau, \infty) \to \mathbb{R}$ is a solution of Eq. (4) with the initial value $\varphi \in X$, if it is a solution and $I_0^\varphi = \varphi$. It follows from (7) that if $\varphi \in X$ with $\varphi(0) > 0$, then $I^\varphi(t) > 0$ for all $t > 0$. Such solutions are called nontrivial solutions. Similarly, if $\varphi \in X_0 := \{\varphi \in X : \varphi(0) = 0\}$, then $I^\varphi(t) = 0$ for all $t \geq 0$.

Note that depending on the choice of I, $\Phi_1(t, I)$ and $\Phi_2(t, I)$ may explode in finite time, but this does not cause any problem in our case as long as solutions of Eq. (4) are in the interval $[0, 1]$.

3. Results on equilibria

We denote by I^* the equilibria (constant solutions) of Eq. (4), and for simplicity we use the same notation for the constant functions as elements of X and their value.

Proposition 3.1. (a) *The disease free equilibrium $I_0^* = 0$ always exists,*
(b) *if $1 < R_0 < \frac{1}{1-p}$, Eq. (4) has a unique positive equilibrium $I_1^* = 1 - \frac{1}{R_0}$,*
(c) *if $R_0 > \frac{1}{q(1-p)}$, Eq. (4) has a different unique positive equilibrium $I_2^* = 1 - \frac{1}{qR_0}$,*
(d) *if $R_0 \leq 1$ or $\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}$, there is no positive equilibrium.*

Proof. To obtain the equilibria of Eq. (4), we set $I(t) \equiv I(t-\tau) \equiv I^*$, and let the right hand side of Eq. (4) be zero, so

$$R_0 h(I^*)I^*(1 - I^*) = I^*. \tag{8}$$

$I^* = 0$ is a solution, which corresponds to the disease free equilibrium. If $I^* \neq 0$, then from Eq. (8) we get

$$R_0 h(I^*)(1 - I^*) = 1. \tag{9}$$

This equation does not have a positive solution for $R_0 \leq 1$. If $R_0 > 1$, we distinguish two cases. If $I^* < p$, then $I^* = 1 - \frac{1}{R_0}$ (let us denote it by I_1^*), and if $I^* \geq p$, then we have $I^* = 1 - \frac{1}{qR_0}$ (denoted by I_2^*). To satisfy $I_1^* < p$, we need $1 - \frac{1}{R_0} < p$, which is equivalent to $R_0 < \frac{1}{1-p}$. Similarly, $I_2^* > p$ is equivalent to $R_0 > \frac{1}{q(1-p)}$. The case $I_2^* = p$ corresponds to $R_0 = \frac{1}{q(1-p)}$ which has been excluded. □

Proposition 3.2. (a) *If $R_0 \leq 1$, then I_0^* is globally asymptotically stable. If $R_0 > 1$, then I_0^* is unstable.*
(b) *If $1 < R_0 < \frac{1}{1-p}$, the positive equilibrium I_1^* is globally asymptotically stable on $X \setminus X_0$.*
(c) *If $R_0 > \frac{1}{q(1-p)}$, the positive equilibrium I_2^* is globally asymptotically stable on $X \setminus X_0$.*

Proof. If $R_0 \leq 1$, statement (a) easily follows from the comparison

$$\frac{dI(t)}{dt} = [R_0 h(I(t-\tau)) - 1]I(t) - R_0 h(I(t-\tau))I(t)^2 \leq -qR_0 I(t)^2. \tag{10}$$

As

$$\frac{dI(t)}{dt} = I(t)[R_0 - 1] \tag{11}$$

is the linear variational equation around zero, the disease free equilibrium is unstable if $R_0 > 1$.

Now suppose that $1 < R_0 < \frac{1}{1-p}$. In order to prove (b), first we show that for all solutions $I(t)$,

$$I^\infty = \limsup_{t\to\infty} I(t) \leq I_1^*.$$

Indeed, if $I(t) > I_1^* = 1 - \frac{1}{R_0}$ for some $t \geq 0$, then

$$R_0 h(I(t-\tau))(1 - I(t)) < 1,$$

and

$$I'(t) = I(t)(-1 + R_0 h(I(t-\tau))(1 - I(t))) < 0.$$

Thus $I(t)$ is strictly decreasing at t. This reasoning shows that if $I(t_0) \leq I_1^*$ for some $t_0 \geq 0$, then $I(t) \leq I_1^*$ for all $t \geq t_0$. On the other hand, if $I(t) > I_1^*$ for all $t \geq 0$, then $I(t)$ is strictly decreasing on $[0, \infty)$, and $I^* = \lim_{t\to\infty} I(t)$ exists. In this case Eq. (4) implies that $\lim_{t\to\infty} I'(t)$ also exists. As $I(t)$ is strictly decreasing and bounded from below, necessarily $\lim_{t\to\infty} I'(t) = 0$. Thus $0 = I^*(-1 + R_0 h(I^*)(1 - I^*))$ and I^* is an equilibrium. As there are no equilibria greater than I_1^*, we have $I^* = I_1^*$. Summing up, $I^\infty \leq I_1^*$ for all solutions $I(t)$ of Eq. (4).

Hence for all nontrivial solutions $I(t)$, $t_0 \geq 0$ can be given such that $I(t) < p$ for all $t > t_0 - \tau$ and $I(t)$ satisfies the ordinary differential equation

$$\frac{dI(t)}{dt} = -I(t) + R_0 I(t)(1 - I(t)) \tag{12}$$

on (t_0, ∞). Solving (12), we obtain that

$$I(t) = \frac{I(t_0)(R_0 - 1)}{I(t_0)R_0 + [(1 - I(t_0))R_0 - 1]e^{-(R_0-1)(t-t_0)}} \quad \text{for } t > t_0.$$

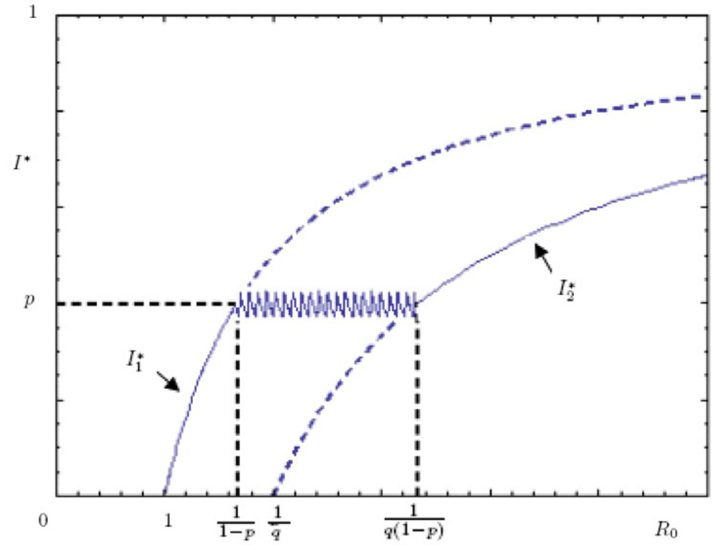

Fig. 1. The relation of I^* and R_0. The solutions oscillate when $\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}$.

As $I(t_0) \neq 0$, it follows that $\lim_{t\to\infty} I(t) = 1 - \frac{1}{R_0} = I_1^*$.

We can prove (c) by using a similar argument, by showing that

$$I_\infty = \liminf_{t\to\infty} I(t) \geq I_2^* > p$$

for all nontrivial solutions $I(t)$, and by verifying that the solutions of the ordinary differential equation

$$\frac{dI(t)}{dt} = -I(t) + qR_0 I(t)(1 - I(t))$$

converge to I_2^* as $t \to \infty$. □

The graph showing the relation of equilibria I^* and R_0 is depicted in Fig. 1. From Fig. 1, we can also see the effect of the network structure. By increasing the average degree $\langle k\rangle$, R_0 is also increasing, and the equilibrium will change from zero to nonzero, then to oscillations (to be proven in the next section), then back to a nonzero equilibrium again; thus system (4) will experience different dynamical behaviors.

4. Oscillation results

From now on we assume that

$$\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}. \tag{13}$$

Note that if $I(t_1) = p$ for some $t_1 \geq 0$, then $I'(t_1) \neq 0$. Indeed, $I'(t_1) = \frac{\partial}{\partial t}\Phi_i(0, p)$ with some $i \in \{1, 2\}$, which is nonzero under the assumption (13). It follows that for all $\varphi \in X$ and $t \geq 0$, the measure of the set $\{\theta \in [-\tau, 0] : I_t^\varphi(\theta) = p\}$ is zero. This observation is used in the subsequent proof.

4.1. Oscillation around p

Proposition 4.1. *If (13) holds, then all nontrivial solutions of Eq. (4) oscillate around p.*

Proof. Suppose for contradiction that either there exists T_1 such that $I(t) \geq p$ for all $t > T_1$, or a T_2 can be given so that $I(t) \leq p$ for all $t > T_2$ and $I(t)$ is a nontrivial solution.

If $I(t) \geq p$ for all $t > T_1$, then

$$I'(t) = I(t)(-1 + R_0 q(1 - I(t))) < I(t)(-1 + R_0 q(1 - p)) < 0$$

for all $t > T_1 + \tau$, and $I^* = \lim_{t\to\infty} I(t) \geq p$ exists. It is easy to see that $\lim_{t\to\infty} I'(t) = 0$ in this case, and $0 = I^*(-1+R_0 q(1-I^*))$, i.e. $I^* = 1 - \frac{1}{qR_0}$ is an equilibrium. This is a contradiction, as there are no positive equilibria for (13).

If $I(t) < p$ for all $t > T_2$, then

$$I'(t) = I(t)(-1 + R_0(1 - I(t))) > I(t)(-1 + R_0(1 - p)) > 0$$

for all $t > T_1 + \tau$ with $I(t) \neq 0$. Hence if $I(t)$ is a nontrivial solution, then $I(t)$ converges to the equilibrium $I^* = 1 - \frac{1}{R_0} > 0$ in a similar way, which is a contradiction again.

We finish by demonstrating that if $I(t) \leq p$ for all $t > T_2$, then a threshold number T_3 can be given such that $I(t) < p$ for all $t > T_3$. If that is not true, then we can find arbitrarily large t_k such that $I(t_k) = p$. If $I(t_k - \tau) < p$, then I' is continuous at t_k and $I'(t_k) = 0$ must hold, but then $h(I(t_k - \tau)) = 1$ and so

$$I'(t_k) = p(-1 + R_0(1 - p)) \neq 0.$$

If $I(t_k - \tau) = p$, then by integration we obtain that

$$1 = \frac{I(t_k)}{I(t_k - \tau)} = \exp\left\{\int_{t_k-\tau}^{t_k} [-1 + R_0 h(I(s - \tau))(1 - I(s))]ds\right\},$$

which implies that the integral in the exponent should be zero. However, $R_0 h(I(s - \tau))(1 - I(s)) - 1 \geq R_0(1 - p) - 1 > 0$ whenever $I(s - \tau) < p$, and the equality $I(s - \tau) = p$ holds on a set of measure zero. Thus the integral is positive, which is a contradiction. □

We introduce a discrete Lyapunov functional analogous to the one given by Mallet-Paret and Sell in [23].

For $\varphi \in C$, set $sc(\varphi) = 0$ if either $\varphi(s) \geq p$ for all $s \in [-\tau, 0]$, or $\varphi(s) \leq p$ for all $s \in [-\tau, 0]$. Otherwise define

$$sc(\varphi) = \sup\Big\{k \in \mathbb{N} \setminus \{0\} : \text{ there exists a strictly increasing sequence } (s_i)_0^k \subseteq [-\tau, 0] \text{ with } (\varphi(s_{i-1}) - p)(\varphi(s_i) - p) < 0 \text{ for } i \in \{1, 2, \ldots, k\}\Big\}.$$

Then define $V : C \to \mathbb{N} \cup \{\infty\}$ by

$$V(\varphi) = \begin{cases} sc(\varphi), & \text{if } sc(\varphi) \text{ is odd,} \\ sc(\varphi) + 1, & \text{if } sc(\varphi) \text{ is even.} \end{cases}$$

V has the following lower semi-continuity property (for a proof, see [23]): For each $\varphi \in C$ and $(\varphi_n)_0^\infty \subset C$ with $\varphi_n \to \varphi$ as $n \to \infty$, $V(\varphi) \leq \liminf_{n\to\infty} V(\varphi_n)$.

The next lemma shows that V is indeed a Lyapunov functional and therefore X is invariant under the condition (13). The proof is a slight modification of the proof of Lemma VI.2 in [24].

Lemma 4.2. *If (13) holds, then $t \mapsto V(I_t)$ is monotone nonincreasing for any solution $I(t)$ of Eq. (4).*

Proof. Let $t_2 > t_1 \geq 0$ be arbitrary. We need to verify that $V(I_{t_1}) \geq V(I_{t_2})$.

We claim that it suffices to show that for all $t \geq 0$, there exists $\varepsilon_0 = \varepsilon_0(t) > 0$ so that $V(I_t) \geq V(I_{t+\varepsilon})$ for all $\varepsilon \in [0, \varepsilon_0]$. Suppose this property holds. Set

$$t^* = \sup\{t \in [t_1, t_2] : V(I_{t_1}) \geq V(I_u) \text{ for all } t_1 \leq u \leq t\}.$$

Clearly $t_1 \leq t^* \leq t_2$. By definition, there exists a sequence $(s_n)_{n=0}^\infty$ in $[t_1, t^*]$ such that $s_n \to t^*$ as $n \to \infty$ and $V(I_{t_1}) \geq V(I_{s_n})$ for all $n \geq 0$. Since $I_{s_n} \to I_{t^*}$ as $n \to \infty$, the lower semi-continuity property of V implies that $V(I_{t_1}) \geq V(I_{t^*})$. If $t^* < t_2$, then a threshold number $\varepsilon_0(t^*) \in (0, t_2 - t^*)$ can be given so that $V(I_{t_1}) \geq V(I_{t^*}) \geq V(I_{t^*+\varepsilon})$ for all $\varepsilon \in [0, \varepsilon_0(t^*)]$. This contradicts the choice of t^*. So $V(I_{t_1}) \geq V(I_{t_2})$, and the claim holds.

We confirm that for any $t \geq 0$, a threshold number $\varepsilon_0 = \varepsilon_0(t) > 0$ can be given so that $V(I_t) \geq V(I_{t+\varepsilon})$ for all $\varepsilon \in [0, \varepsilon_0]$. The cases $I(t) \neq p$ or $V(I_t) = \infty$ are clear, so assume that $I(t) = p$ and $V(I_t) < \infty$. Then there exists $\varepsilon_0 > 0$ such that either

(i) $I(s) \geq p$ for $s \in [t - \tau, t - \tau + \varepsilon_0]$ and $I(s) - p$ does not change sign on $[t - \varepsilon_0, t]$,

or

(ii) $I(s) < p$ for $s \in (t - \tau, t - \tau + \varepsilon_0]$ and $I(s) - p$ does not change sign on $[t - \varepsilon_0, t]$.

In case (i),

$$I(s) = \begin{cases} \dfrac{p(qR_0 - 1)}{pqR_0 + [(1 - p)qR_0 - 1]e^{-(qR_0-1)(s-t)}}, & \text{if } qR_0 \neq 1, \\ \dfrac{p}{p(s - t) + 1}, & \text{if } qR_0 = 1 \end{cases} \quad \text{for all } s \in [t, t + \varepsilon_0].$$

As $(1-p)qR_0 - 1 < 0$, we see that $I(s)$ is a strictly decreasing function on $[t, t+\varepsilon_0]$ regardless of the value of $qR_0 - 1$. Hence if $I(s) \leq p$ for $s \in [t - \varepsilon_0, t]$, then $I(s) - p$ does not change sign on $[t - \varepsilon_0, t + \varepsilon_0]$, and $V(I_t) \geq V(I_{t+\varepsilon})$ for all $\varepsilon \in [0, \varepsilon_0]$. If $I(s) \geq p$ for $s \in [t - \varepsilon_0, t]$, then $sc(I_t)$ is even and $V(I_t) = sc(I_t) + 1$. As $I(s) - p$ admits at most one sign change on $[t - \varepsilon_0, t + \varepsilon]$ for all $\varepsilon \in [0, \varepsilon_0]$, we conclude that $sc(I_{t+\varepsilon}) \leq sc(I_t) + 1$ and $V(I_{t+\varepsilon}) \leq V(I_t)$ for all $\varepsilon \in [0, \varepsilon_0]$.

In case (ii),

$$I(s) = \frac{p(R_0-1)}{pR_0 + [(1-p)R_0 - 1]e^{-(R_0-1)(s-t)}} \quad \text{for } s \in [t, t+\varepsilon_0].$$

As $R_0 > 1$ and $(1-p)R_0 - 1 > 0$, $I(s)$ is strictly increasing on $[t, t+\varepsilon_0]$. From now on, the proof is analogous to the one given for case (i); therefore we omit the details. □

4.2. Periodic solutions

A solution $I(t)$ of Eq. (4) is called slowly oscillatory, if $V(I_t) = 1$ for all $t \geq 0$, and it is rapidly oscillatory if $V(I_t) > 1$ for all $t \geq 0$.

Theorem 4.3. *Set p, q and R_0 according to (13). Then for each delay $\tau > 0$, there exists a unique slowly oscillatory periodic solution $I^\tau(t)$ of Eq. (4) up to time translation. The minimal period of $I^\tau(t)$ is $T(\tau) = 2\tau + \nu_1(\tau) + \nu_2(\tau)$, where $\nu_1(\tau), \nu_2(\tau) > 0$ are given by*

$$\nu_1(\tau) = \begin{cases} \dfrac{1}{qR_0 - 1} \ln\left(\dfrac{(qR_0-1)\,[(1-p)R_0 - 1]\,e^{-(R_0-1)\tau} - pR_0(1-q)}{(R_0-1)\,[(1-p)qR_0 - 1]}\right), & \text{if } qR_0 \neq 1, \\ \dfrac{[(1-p)R_0 - 1]\left[1 - e^{-(R_0-1)\tau}\right]}{p(R_0-1)}, & \text{if } qR_0 = 1 \end{cases} \tag{14}$$

and

$$\nu_2(\tau) = \begin{cases} \dfrac{1}{R_0 - 1} \ln\left(\dfrac{(R_0-1)\,[(1-p)\,qR_0 - 1]\,e^{-(qR_0-1)\tau} + pR_0\,(1-q)}{(qR_0-1)\,[(1-p)R_0 - 1]}\right), & \text{if } qR_0 \neq 1, \\ \dfrac{1}{R_0 - 1} \ln\left(\dfrac{(p\tau+1)\,(R_0-1) - pR_0}{(1-p)R_0 - 1}\right), & \text{if } qR_0 = 1 \end{cases} \in (0, \infty). \tag{15}$$

$I^\tau(t)$ is stable in the sense of Lyapunov and it is explicitly given by

$$I^\tau(t) = \begin{cases} \Phi_1(t, p) & \text{for } t \in [0, \tau], \\ \Phi_2(t - \tau - \nu_1(\tau), p) & \text{for } t \in (\tau, 2\tau + \nu_1(\tau)], \\ \Phi_1(t - 2\tau - \nu_1(\tau) - \nu_2(\tau), p) & \text{for } t \in (2\tau + \nu_1(\tau), 2\tau + \nu_1(\tau) + \nu_2(\tau)] \end{cases} \tag{16}$$

on $[0, T(\tau)]$. In addition, for all $\varphi \in X$ with $V(\varphi) = 1$, a number $t_1 \geq 0$ and a constant $\xi \in [0, T(\tau))$ can be given such that $I^\varphi(t) = I^\tau(t+\xi)$ for all $t \geq t_1$.

Proof. 1. Consider the set

$$A = \{\varphi \in X : \varphi(s) \leq p \text{ for all } s \in [-\tau, 0], \varphi(0) = p\}$$

of initial functions. Then for any $\psi \in A$, $I(t) = I^\psi(t) = \Phi_1(t, p)$ for all $t \in [0, \tau]$ independently of the specific form of ψ (see Fig. 2). It is easy to see from formula (5) that $I(t)$ strictly increases on $[0, \tau]$; hence $I(t) > p$ for $t \in (0, \tau]$. Set

$$p_1 = \Phi_1(\tau, p) = \frac{p(R_0-1)}{pR_0 + [(1-p)R_0 - 1]e^{-(R_0-1)\tau}} > p. \tag{17}$$

By Proposition 4.1, there exists $\nu_1 = \nu_1(\tau) \in (0, \infty)$ such that $I(t) > p$ on $(\tau, \tau + \nu_1)$ and $I(\tau + \nu_1) = p$. It is also clear that $I(t) = \Phi_2(t - \tau, p_1)$ on $[\tau, 2\tau + \nu_1]$. Hence from the equation $\Phi_2(\nu_1, p_1) = p$ we get

$$\nu_1 = \begin{cases} \dfrac{1}{qR_0 - 1} \ln\left(\dfrac{p[(1-p_1)qR_0 - 1]}{p_1[(1-p)qR_0 - 1]}\right), & \text{if } qR_0 \neq 1, \\ \dfrac{1}{p} - \dfrac{1}{p_1} & \text{otherwise.} \end{cases} \tag{18}$$

For all $t \in [\tau + \nu_1, 2\tau + \nu_1]$, the solution is given by $I(t) = \Phi_2(t - \tau - \nu_1, p)$. Hence formula (6) shows that $I(t)$ is strictly decreasing on $[\tau + \nu_1, 2\tau + \nu_1]$ and $I(t) < p$ for $t \in (\tau + \nu_1, 2\tau + \nu_1]$. Set

$$p_2 = \Phi_2(\tau, p) = \begin{cases} \dfrac{p(qR_0-1)}{pqR_0 + [(1-p)qR_0 - 1]e^{-(qR_0-1)\tau}}, & \text{if } qR_0 \neq 1, \\ \dfrac{p}{p\tau + 1}, & \text{if } qR_0 = 1. \end{cases} \tag{19}$$

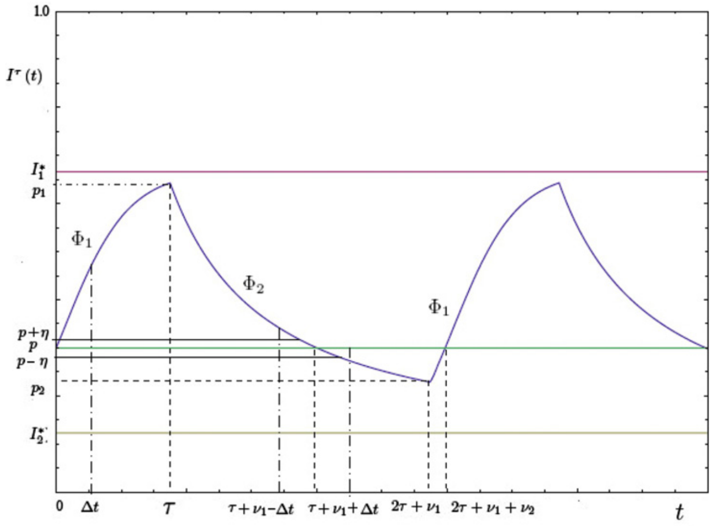

Fig. 2. Periodic solution $I^\tau(t)$.

Then $p_2 \in (0, p)$ and $I(2\tau + \nu_1) = p_2$. We use Proposition 4.1 again to get $\nu_2 = \nu_2(\tau) \in (0, \infty)$ such that $I(t) < p$ on $(2\tau + \nu_1, 2\tau + \nu_1 + \nu_2)$ and $I(2\tau + \nu_1 + \nu_2) = p$. As necessarily $I(t) = \Phi_1(t - 2\tau - \nu_1, p_2)$ on the interval $[2\tau + \nu_1, 2\tau + \nu_1 + \nu_2]$, one can compute that

$$\nu_2 = \frac{1}{R_0 - 1} \ln \left(\frac{p[(1 - p_2)R_0 - 1]}{p_2[(1 - p)R_0 - 1]} \right). \tag{20}$$

We obtained that $I_{2\tau+\nu_1+\nu_2} \in A$. So if $I^\tau(t)$ is defined to be the periodic extension of $I(t)|_{[0,2\tau+\nu_1+\nu_2]}$ to $[-\tau, \infty)$, then $I^\tau(t)$ is a slowly oscillatory periodic solution of Eq. (4) with minimal period $T = 2\tau + \nu_1 + \nu_2$; furthermore it is given by $\Phi_1(t, p)$ on $[0, \tau]$, by $\Phi_2(t-\tau, p_1) = \Phi_2(t-\tau-\nu_1, p)$ on $(\tau, 2\tau+\nu_1]$ and by $\Phi_1(t-2\tau-\nu_1, p_2) = \Phi_1(t-2\tau-\nu_1-\nu_2, p)$ on $(2\tau + \nu_1, 2\tau + \nu_1 + \nu_2]$.

The formulas for ν_1, ν_2 in the theorem are derived by substituting p_1 and p_2 into (18) and (20), respectively.

2. Suppose that $\varphi \in X$ with $V(\varphi) = 1$ and consider the solution I^φ. We show that there is a constant $\xi \in [0, T(\tau))$ such that $I^\varphi(t) = I^\tau(t + \xi)$ for all t large enough. Proposition 4.1 guarantees the existence of $t_0 \geq \tau$ such that $I^\varphi(t_0) = p$. Lemma 4.2 gives that $V(I^\varphi_{t_0}) = 1$, that is $I^\varphi(t) - p$ admits at most one sign change on $[t_0 - \tau, t_0]$.

First assume that $I^\varphi(t) - p$ has no sign change on $[t_0 - \tau, t_0]$. If $I^\varphi_{t_0}(s) \geq p$ for all $s \in [-\tau, 0]$, then $I^\varphi(t) = \Phi_2(t - t_0, p)$ for $t \in [t_0, t_0 + \tau]$, and it is clear from the above construction that $I^\varphi(t_0 + s) = I^\tau(\tau + \nu_1 + s)$ for all $s \geq 0$. If $I^\varphi_{t_0}(s) \leq p$ for all $s \in [-\tau, 0]$, then $I^\varphi_{t_0}$ belongs to function class A, and $I^\varphi(t_0 + s) = I^\tau(s)$ for all $s \geq 0$.

It remains to examine the case when $I^\varphi_{t_0}$ has one sign change, that is $u \in (0, \tau)$ can be given such that $I^\varphi_{t_0}(-u) = p$, $I^\varphi_{t_0}$ is not the constant p function and either

(i) $I^\varphi_{t_0}(s) \leq p$ for all s in $[-\tau, -u]$ and $I^\varphi_{t_0}(s) \geq p$ for all s in $[-u, 0]$,

or

(ii) $I^\varphi_{t_0}(s) \geq p$ for all $s \in [-\tau, -u]$ and $I^\varphi_{t_0}(s) \leq p$ for all $s \in [-u, 0]$.

In case (i), $I^\varphi_{t_0}(s) < p$ for all s in $[-\tau, -u]$ except for a finite number of points; hence $I^\varphi(t) = \Phi_1(t - t_0, p)$ for all $t \in [t_0, t_0 + \tau - u]$. It follows that I^φ is strictly increasing on $[t_0, t_0 + \tau - u]$, and $I^\varphi(t) \geq p$ for all $t \in [t_0 - u, t_0 + \tau - u]$. Let $t_0' \geq t_0 + \tau - u$ be minimal with $I^\varphi(t_0') = p$. Then $I^\varphi(t_0' + s) = I^\tau(\tau + \nu_1 + s)$ for all $s \geq 0$. The proof is analogous in case (ii).

The uniqueness of the slowly oscillatory periodic solution follows immediately up to time translation.

3. At last we confirm that to all $\varepsilon > 0$, there corresponds $\delta = \delta(\varepsilon) > 0$ such that if $\varphi \in X$ and $\left\| \varphi - I_0^\tau \right\| < \delta$, then $|I^\varphi(t) - I^\tau(t)| < \varepsilon$ for all $t \geq 0$.

First note that the periodic solution $I^\tau(t)$ is Lipschitz continuous because it is piecewise continuously differentiable. Let $\alpha > 0$ be a Lipschitz constant for $I^\tau(t)$ and let

$$\Delta t \in \left(0, \min\left\{\frac{\varepsilon}{\alpha}, \frac{\nu_1}{2}, \tau\right\}\right).$$

Then $|I^\tau(t + \Delta t) - I^\tau(t)| \leq \alpha \Delta t < \varepsilon$ for all $t \geq -\tau$. Set $\eta > 0$ with

$$\eta < \min\left\{I^\tau(\Delta t) - p, I^\tau(\tau + \nu_1 - \Delta t) - p, p - I^\tau(\tau + \nu_1 + \Delta t), \varepsilon\right\}.$$

This is possible as the expressions on the right hand side are positive. As $I^\tau(t)$ is strictly increasing on $[0, \tau]$, strictly decreasing on $[\tau, \tau+\nu_1]$ and $0 < \Delta t < \tau < \tau+\nu_1-\Delta t < \tau+\nu_1$, we see that $I^\tau(t) > p+\eta$ on $[\Delta t, \tau+\nu_1-\Delta t]$ (see Fig. 2). Note that the length of this interval is greater than τ.

As the solutions of Eq. (4) depend continuously on the initial function, there exists $\delta > 0$ so that if $\varphi \in X$ and $\|\varphi - I_0^\tau\| < \delta$, then $|I^\varphi(t) - I^\tau(t)| < \eta$ for all $t \in [0, T]$. We show that $|I^\varphi(t) - I^\tau(t)| < \varepsilon$ holds for all $t \geq 0$.

On the one hand, $I^\varphi(t) > I^\tau(t) - \eta > p$ on $[\Delta t, \tau+\nu_1-\Delta t]$, which interval has length greater than τ. On the other hand, $I^\varphi(\tau+\nu_1+\Delta t) < I^\tau(\tau+\nu_1+\Delta t)+\eta < p$. So there exists $r \in (\tau+\nu_1-\Delta t, \tau+\nu_1+\Delta t)$ such that $I^\varphi(r) = p$ and $I^\varphi(t) \geq p$ for $t \in [r-\tau, r]$. Consequently $I^\varphi(t) = I^\tau(t+\tau+\nu_1-r)$ and

$$|I^\varphi(t) - I^\tau(t)| = |I^\tau(t+\tau+\nu_1-r) - I^\tau(t)| \leq \alpha\Delta t < \varepsilon$$

for all $t \geq r$. The choice of η implies that $|I^\varphi(t) - I^\tau(t)| < \varepsilon$ also for $t \in [0, r)$. Hence $I^\tau(t)$ is stable in the sense of Lyapunov. □

Theorem 4.4. *If p, q and R_0 satisfy (13), then for each $\tau > 0$ and $k \geq 1$, Eq. (4) admits a rapidly oscillatory periodic solution with segments in $V^{-1}(2k+1)$.*

Proof. Our construction is similar to the one in [25]. By Theorem 4.3, $\nu_i(\tau)$ is a positive continuous function of τ with $\lim_{\tau\to0+}\nu_i(\tau) = 0$ for both $i \in \{1, 2\}$. Hence $T(\tau) = 2\tau+\nu_1(\tau)+\nu_2(\tau)$ is continuous, $\lim_{\tau\to0+}T(\tau) = 0$ and $\lim_{\tau\to\infty}T(\tau) = \infty$. It follows that for all $\tau > 0$ and $k \geq 1$, the equation $\tau = \tau' + kT(\tau')$ admits at least one solution τ'. The slowly oscillatory solution $I^{\tau'}$ given by Theorem 4.3 satisfies

$$\frac{dI(t)}{dt} = -I(t) + R_0 h(I(t-\tau'-kT(\tau')))I(t)(1-I(t)),$$

that is $I^{\tau'}(t)$ is a periodic solution also for the delay $\tau = \tau'+kT(\tau')$. It is easy to see from (16) that $I^{\tau'}(t)-p$ has $2k$ or $2k+1$ sign changes on each interval of length τ. Hence the proof is complete. □

4.3. Permanence

We say that the disease is permanent, if there is a positive constant such that for sufficiently large t, every nonzero solution $I(t)$ is bounded by this constant from below, independently from initial data. It is clear from Proposition 3.2 that Eq. (4) is permanent if $1 < R_0 < \frac{1}{1-p}$ or if $R_0 > \frac{1}{q(1-p)}$. We have an analogous result for the case (13).

Proposition 4.5. *If (13) holds, then for all nontrivial solutions $I(t)$,*

$$0 < p_2 \leq \liminf_{t\to\infty} I(t) \leq \limsup_{t\to\infty} I(t) \leq p_1,$$

where p_1 and p_2 are defined by (17) and (19), respectively.

Proof. Proposition 4.1 implies that for all nontrivial solutions $I(t)$, there exists $t_0 \geq 0$ with $I(t_0) = p$. In the following we prove that $p_2 \leq I(t) \leq p_1$ for all $t \geq t_0$.

1. Suppose for contradiction that there is $\varepsilon > 0$ such that $I(t) = p_2 - \varepsilon$ for some $t \geq t_0$, and let $t_1 > t_0$ be minimal with $I(t_1) = p_2 - \varepsilon$. Then $I'(t_1) \leq 0$. It follows that

$$I(t_1)[-1 + R_0 h(I(t_1-\tau))(1-I(t_1))] \leq 0,$$

that is $R_0 h(I(t_1-\tau))(1-I(t_1)) \leq 1$. With $I(t_1-\tau) < p$ we would have

$$R_0 h(I(t_1-\tau))(1-I(t_1)) > R_0(1-p) > 1$$

by (13). So $I(t_1-\tau) \geq p$. Hence if $t_0' \in [t_0, t_1)$ is chosen to be maximal with $I(t_0') = p$, then $t_1 - t_0' \leq \tau$.

As $I'(t) \geq -I(t) + R_0 q I(t)(1-I(t))$ for all $t \geq t_0'$ and $t \mapsto \Phi_2(t, p)$ is strictly decreasing, we have the estimate

$$I(t) \geq \Phi_2(t-t_0', p) \geq \Phi_2(\tau, p) = p_2$$

for all $t \in \left[t_0', t_0'+\tau\right]$. This contradicts our initial assumption $I(t_1) = p_2 - \varepsilon$ because $t_1 \in [t_0', t_0'+\tau]$.

2. If $I(t) = p_1 + \varepsilon$ for some $t \geq t_0$ and $\varepsilon > 0$, set $t_1 > t_0$ to be minimal with $I(t_1) = p_1 + \varepsilon$. Then

$$I(t_1)[-1 + R_0 h(I(t_1-\tau))(1-I(t_1))] = I'(t_1) \geq 0$$

and $R_0 h(I(t_1-\tau))(1-I(t_1)) \geq 1$. It follows that $I(t_1-\tau) < p$. Indeed, inequality $I(t_1-\tau) \geq p$ would imply $R_0 h(I(t_1-\tau))(1-I(t_1)) < R_0 q(1-p) < 1$. As in the previous case, we define $t_0' \in [t_0, t_1)$ to be maximal with $I(t_0') = p$. Then $t_1 - t_0' \leq \tau$ and we have the estimate

$$I(t_1) \leq \Phi_1(t_1-t_0', p) \leq \Phi_1(\tau, p) = p_1$$

contradicting the assumption that $I(t_1) = p_1 + \varepsilon$.

So $p_2 \leq I(t) \leq p_1$ for all $t \geq t_0$. □

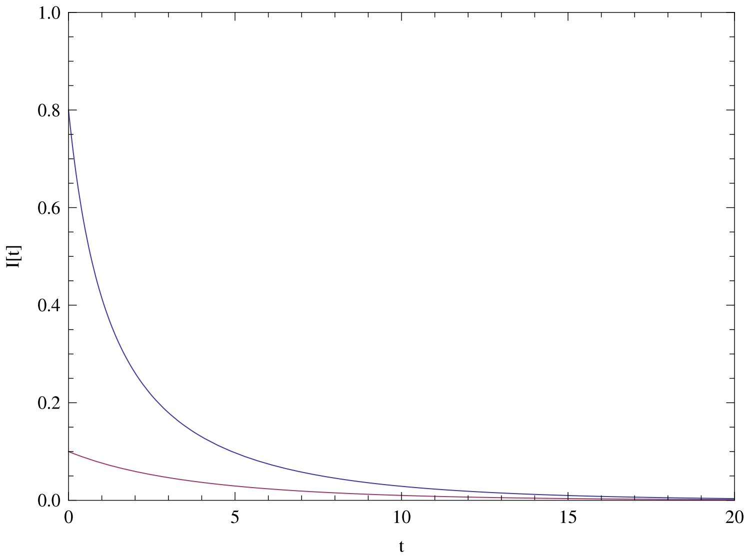

Fig. 3. The time evolutions of the density of infection with $\varphi(t) = 0.1$ and 0.8, $t \in [-\tau, 0]$, $R_0 = 0.8$, $p = 0.8$, $q = 0.6$, and $\tau = 2$.

Note that $p_1 \to p$, $p_2 \to p$ as $\tau \to 0+$ and $p_1 \to 1 - 1/R_0$,

$$p_2 \to \begin{cases} 1 - \dfrac{1}{qR_0}, & \text{if } qR_0 > 1, \\ 0, & \text{if } qR_0 \le 1 \end{cases}$$

as $\tau \to \infty$.

5. Simulations

In this section, we discuss some examples and simulations. Our purpose is to illustrate the sharpness of the results of the previous section. First we set initial data as constant functions. Let $p = 0.8$, $q = 0.6$ and demonstrate the stability of the zero equilibrium, as shown in Fig. 3 with $R_0 = 0.8$.

In the case $R_0 = 2$, as shown in Fig. 4, the positive equilibrium I_1^* is asymptotically stable. While in the case $R_0 = 2.7$, $p = 0.1$, $q = 0.6$, as shown in Fig. 5, the positive equilibrium I_2^* is asymptotically stable.

If $\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}$, then all solutions of Eq. (4) oscillate around p. In this case, to illustrate the effect of the time delay, we consider distinct values of the delay while other parameters are fixed. In the first case the delay is small, depicted in Fig. 6 (left), with $\tau = 0.64$, and the amplitude of the solution around p is apparently small. In the second case, if the delay is larger ($\tau = 3.6$), then we observe increased amplitude in Fig. 6 (right). However, for all delays, the solutions always oscillate between $1 - \frac{1}{R_0}$ and $1 - \frac{1}{qR_0}$. These values are represented by the straight lines in Fig. 6, and one can see that these bounds are rather sharp for large delays.

At last, we also consider different initial functions while the parameters p, q, R_0 are fixed. In the first case the initial function is $\varphi(t) = 0.3 + \sin(3(t-1))$, depicted in Fig. 7 (left). In the second case, the initial function is $\varphi(t) = 0.3 * (0.9 - 0.9 * \sin(10 * (t-1)))$, depicted in Fig. 7 (right). From the figure one can see how solutions with different initial functions tend to the slowly oscillatory periodic solution for $\frac{1}{1-p} < R_0 < \frac{1}{q(1-p)}$.

6. Conclusion

In this paper, we studied a time delay model for an SIS epidemic process on a homogeneous network. We assumed that individuals temporarily reduce the number of their links by a factor q when the density of infections reaches the threshold number p, but this modification in the contact pattern is done with some delay τ. When the basic reproduction number is smaller than or equal to one, the disease will be eradicated. For reproduction numbers larger than one, we showed that the disease persists in the population. In this case the following behaviors are possible:

(i) all links remain active (if the density of infected nodes remain lower than the threshold), the reduction of contacts will never be triggered and the solutions converge to the endemic equilibrium I_1^*;

(ii) the density of infected nodes exceeds the threshold, the reduction of links is triggered, but the density of infected nodes remain above threshold even with the reduced number of links, hence the terminated links remain inactive for all future time and the solutions converge to a different endemic equilibrium I_2^*;

(iii) there is an interesting intermediate situation for a range of basic reproduction numbers, when the reduction is

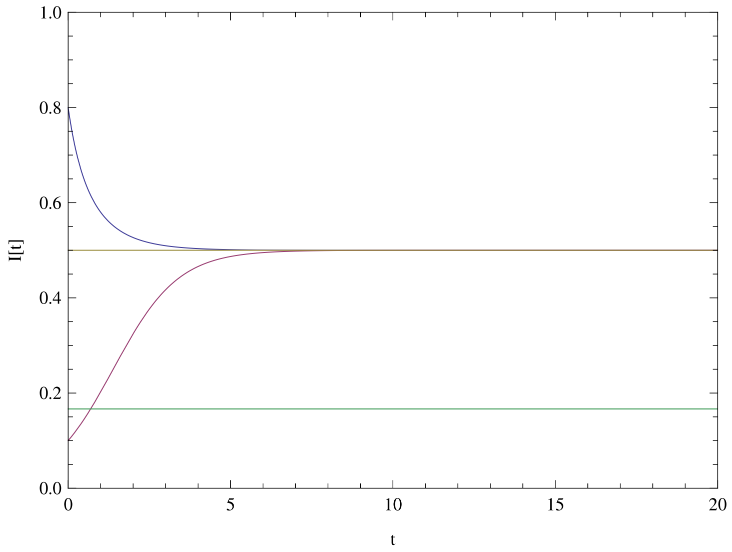

Fig. 4. The time evolutions of the density of infection with $\varphi(t) = 0.1$ and 0.8, $t \in [-\tau, 0]$, $R_0 = 2$, $p = 0.8$, $q = 0.6$ and $\tau = 2$.

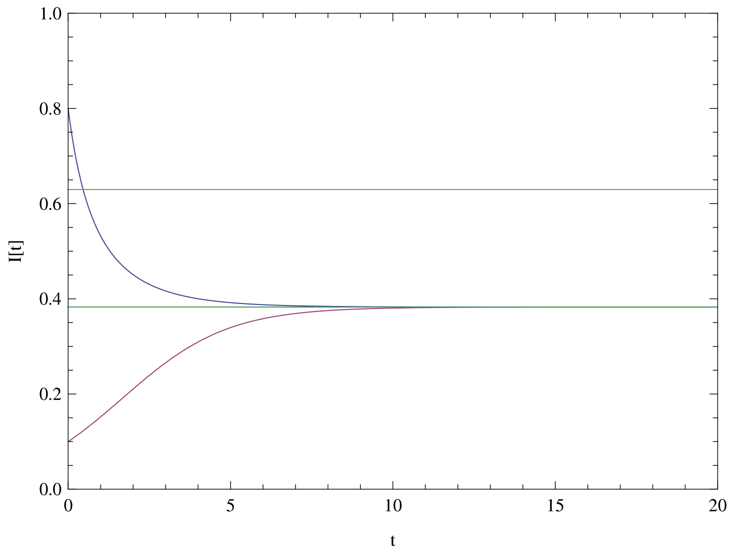

Fig. 5. Evolutions of the density of infection with $\varphi(t) = 0.1$ and 0.8, $t \in [-\tau, 0]$, $R_0 = 2.7$, $p = 0.1$, $q = 0.6$ and $\tau = 2$.

triggered, but with such reduced transmission the density of infected nodes decreases below the threshold; thus the links will be reactivated again, which helps the disease to spread more, thus triggering the reduction, and so on, forming an interesting periodic oscillatory pattern.

We showed that in this regime (iii) all solutions oscillate, and their frequency cannot increase. There is a unique slowly oscillatory solution, which can be computed explicitly and which attracts every other slowly oscillatory solutions (in fact each slowly oscillatory solution jumps onto the same orbit in finite time). However, rapidly oscillatory periodic solutions exist as well for any delay and any frequency. The time delay has significance in determining the characteristics of the slow oscillation: longer delay leads to larger amplitudes. It is typical for delay differential equations that increasing the delay leads to oscillations. In our case, the oscillatory regime is determined not by the delay, but by the reproduction number, which is proportional to the average degree of the network. Thus our results can be interpreted as follows: the solutions converge if the link density of the network is small or large, but there is an intermediate interval when the density of infected nodes in the network oscillates for all future time. Our results indicate that the structure of the network, the switching type reduction in contacts and the delayed decision in reduction interestingly interplay on influencing the spreading dynamics of infectious diseases.

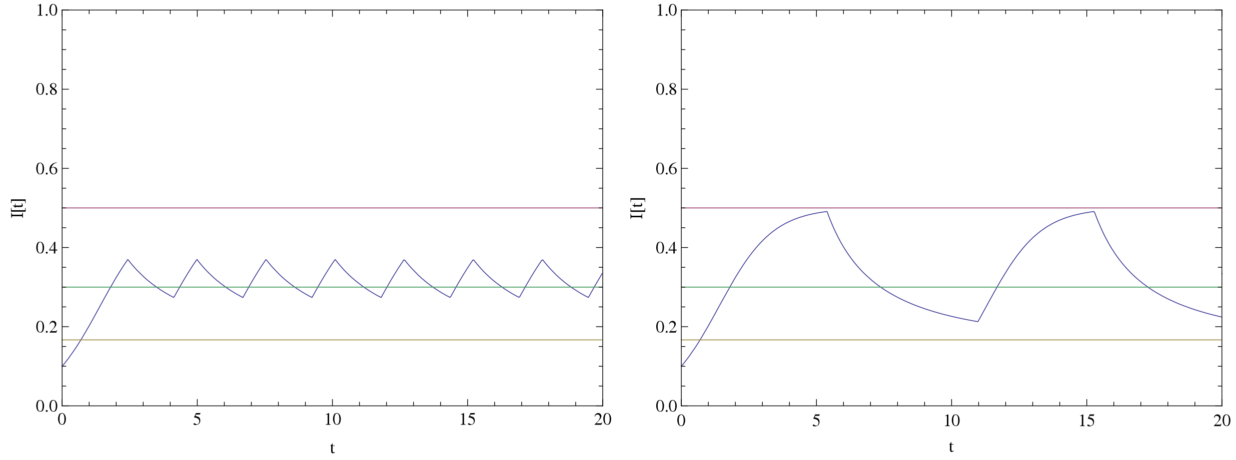

Fig. 6. The time evolutions of the density of infection with $\varphi(t) = 0.1$, $t \in [-\tau, 0]$, $R_0 = 2$, $p = 0.3$, $q = 0.6$ and different delays $\tau = 0.64$ (left) and $\tau = 3.6$ (right).

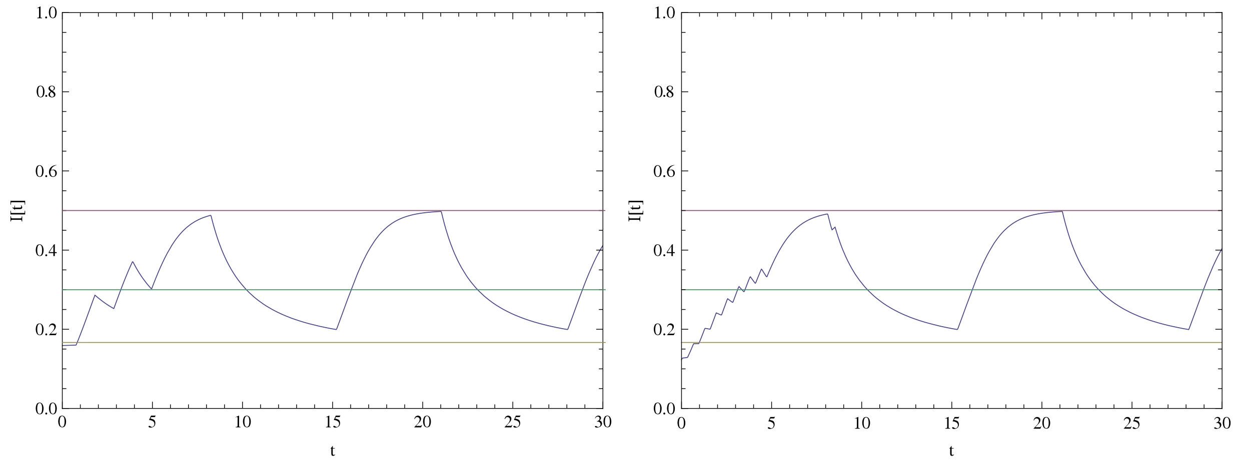

Fig. 7. The time evolutions of the density of infection with $R_0 = 2$, $p = 0.3$, $q = 0.6$, $\tau = 5$ and with different initial functions. The figure on the left is generated with $\varphi(t) = 0.3 + \sin(3(t-1))$, and the figure on the right is generated using $\varphi(t) = 0.3 * (0.9 - 0.9 * \sin(10 * (t-1)))$.

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