1.

$$C^T + 3C = \begin{bmatrix}1 & 2\\-5 & 3\end{bmatrix}^T + 3\cdot\begin{bmatrix}1 & 2\\-5 & 3\end{bmatrix} = \begin{bmatrix}1 & -5\\2 & 3\end{bmatrix} + \begin{bmatrix}3 & 6\\-15 & 9\end{bmatrix} = \underline{\underline{\begin{bmatrix}4 & 1\\-13 & 12\end{bmatrix}}}$$

$A(B^T + C)$ is <u>not defined</u>, because $\underset{2\times 3}{B^T} + \underset{2\times 2}{C}$ is not defined.

$$B + A = \begin{bmatrix}3 & 1\\0 & -2\\-3 & 2\end{bmatrix} + \begin{bmatrix}2 & -2\\1 & 3\\4 & -1\end{bmatrix} = \begin{bmatrix}5 & -1\\1 & 1\\1 & 1\end{bmatrix},$$ so

$$(B+A)C = \underline{\underline{\begin{bmatrix}10 & 7\\-4 & 5\\-4 & 5\end{bmatrix}}}$$

$$\begin{array}{cc} & \underset{C}{\begin{bmatrix}1 & 2\\-5 & 3\end{bmatrix}} \\ \underset{B+A}{\begin{bmatrix}5 & -1\\1 & 1\\1 & 1\end{bmatrix}} & \underset{(B+A)C}{\begin{bmatrix}10 & 7\\-4 & 5\\-4 & 5\end{bmatrix}}\end{array}$$

2.

$$\begin{vmatrix}2 & 1^* & -2 & 1\\1 & 0 & 2 & 4\\3 & 3 & -2 & 0\\2 & 0 & -1 & 3\end{vmatrix} \; -3\cdot \; = \begin{vmatrix}2^+ & 1^- & -2 & 1\\1 & 0 & 2 & 4\\-3 & 0 & 4 & -3\\2 & 0 & -1 & 3\end{vmatrix} = \text{(expand along the 2nd column)}$$

$$= (-1)\cdot 1\cdot \begin{vmatrix}1^* & 2 & 4\\-3 & 4 & -3\\2 & -1 & 3\end{vmatrix} \; \begin{matrix}+3\cdot\\-2\cdot\end{matrix} \; = (-1)\cdot\begin{vmatrix}1^+ & 2 & 4\\0 & 10 & 9\\0 & -5 & -5\end{vmatrix}$$ expand along the 1st col.

$$= (-1)\cdot \underbrace{\begin{vmatrix}10 & 9\\-5 & -5\end{vmatrix}}_{-50-(-45)=-5} = (-1)(-5) = \underline{\underline{5}}$$

3. $\begin{cases} 7x_1 + 9x_2 = -1 \\ 3x_1 + 4x_2 = 5 \end{cases} \longrightarrow |A| = \begin{vmatrix} 7 & 9 \\ 3 & 4 \end{vmatrix} = 28 - 27 = 1$

non-zero, Cramer's rule can be applied.

$$|A^{(1)}| = \begin{vmatrix} -1 & 9 \\ 5 & 4 \end{vmatrix} = -4 - 45 = -49$$

$$\Downarrow$$

$$\underline{\underline{x_1}} = \frac{|A^{(1)}|}{|A|} = \frac{-49}{1} = \underline{\underline{-49}}.$$

$$|A^{(2)}| = \begin{vmatrix} 7 & -1 \\ 3 & 5 \end{vmatrix} = 35 - (-3) = 38$$

$$\Downarrow$$

$$\underline{\underline{x_2}} = \frac{|A^{(2)}|}{|A|} = \frac{38}{1} = \underline{\underline{38}}.$$

the solution is $\boxed{(-49, 38)}$.

4.)

$$\left[\begin{array}{ccc|c} 1^* & -2 & 1 & 0 \\ -1 & 3 & 2 & -1 \\ 2 & 1 & 17 & -5 \end{array}\right] \begin{array}{l} \\ + \\ -2\cdot \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1^* & 3 & -1 \\ 0 & 5 & 15 & -5 \end{array}\right] \begin{array}{l} \\ \\ -5\cdot \end{array} \sim$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Row-echelon form

Basic variables : x_1, x_2.

Free variable : x_3

(2nd) $x_2 + 3x_3 = -1 \longrightarrow x_2 = -1 - 3x_3$

(1st) $x_1 - 2x_2 + x_3 = 0$

$$x_1 = 2x_2 - x_3 = 2(-1-3x_3) - x_3$$
$$= -2 - 6x_3 - x_3 = -2 - 7x_3$$

The free variable can be an arbitrary real number : $x_3 = a$, where 'a' is an arbitrary number

So there are infinitely many solutions:

$\boxed{(-2-7a,\ -1-3a,\ a)}$, where 'a' is an arbitrary number.

5.

$$\left[\begin{array}{ccc|c} -1^* & 1 & -1 & 1 \\ 2 & -3 & 0 & 2 \\ 1 & 2 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & -1^* & -2 & 4 \\ 0 & 3 & 6 & 4 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 0 & 16 \end{array}\right] \leftarrow$$ contradicting row, the linear system has no solutions.

$\Downarrow$

The answer is NO, $s \notin \mathrm{Span}(u, v, w)$.

6.

$$\left[\begin{array}{ccc|c} 1^* & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1^* & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2^* & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$ Row-echelon form

(3rd) $2x_3 = 0 \longrightarrow \underline{x_3 = 0}$

(2nd) $x_2 + x_3 = 0 \longrightarrow \underline{x_2} = -x_3 = \underline{0}$

(1st) $x_1 + 2x_2 + x_3 = 0 \longrightarrow x_1 = -2x_2 - x_3 = -2\cdot 0 - 0 = \underline{0}$

So $(0,0,0)$ is the only solution to the linear system, thus u, v, w are <u>linearly independent</u>.