Powers of matrices

Linear algebra

Lecture 13

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Definition. The k'th power of a square matrix A is defined as repeated multiplication:

$$A^k := \overbrace{A \cdot A \cdot \ldots \cdot A}^{k \text{ times}}.$$

Example.

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix}^3 = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & -4 \\ 18 & 21 & 3 \\ 15 & 30 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -40 & -17 \\ 99 & 99 & 9 \\ 120 & 165 & 33 \end{bmatrix}.$$

Remark. In real-life examples, it is often needed to calculate the powers of a matrix, e.g. see Example 7.10 in the lecture notes.

We will just see that it is trivial to calculate the powers of diagonal matrices.

Notation. We will denote by $\text{Diag}(d_1, \dots, d_n)$ the $n \times n$ diagonal matrix whose diagonal entries are $d_1, \dots, d_n$ (from top to bottom). For example,

$$\text{Diag}(2, 5, -3) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Proposition 1. If $D = \text{Diag}(d_1, \dots, d_n)$, then $D^k = \text{Diag}(d_1^k, \dots, d_n^k)$.
In words, the k'th power of a diagonal matrix D is also a diagonal matrix (of the same size), whose diagonal entries are the k'th power of the diagonal entries of D.

Example.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & -27 \end{bmatrix}.$$

Check this by hand!

Now we consider a much wider class of matrices:

Definition. We say that an $n \times n$ matrix A is diagonalizable if there exist an $n \times n$ diagonal matrix D and an invertible $n \times n$ matrix S such that $A = SDS^{-1}$. (We note that not all square matrices are diagonalizable.)

Proposition 2. Assume that the $n \times n$ matrix A is diagonalizable: $A = SDS^{-1}$. Then

$$A^k = SD^kS^{-1}.$$

By Proposition 1, the matrix D^k can be easily calculated: If $D = \mathrm{Diag}(d_1, \ldots, d_n)$, then $D^k = \mathrm{Diag}(d_1^k, \ldots, d_n^k)$.

Proof.

$$\begin{aligned} A^k = A \cdot A \cdot A \cdot \ldots \cdot A &= (SDS^{-1}) \cdot (SDS^{-1}) \cdot (SDS^{-1}) \cdot \ldots \cdot (SDS^{-1}) \\ &= SDS^{-1}SDS^{-1}SDS^{-1} \cdots SDS^{-1} = SDIDIDI \cdots IDS^{-1} \\ &= SDDD \cdots DS^{-1} = SD^kS^{-1}, \end{aligned}$$

where I is the $n \times n$ identity matrix ($S^{-1}S = I$, by the definition of the inverse matrix), and so $DI = D$. $\square$

Theorem. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable as follows. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A with associated eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (that is, $\mathbf{v}_i$ is an eigenvector of A that corresponds to the eigenvalue λ_i). Let $D = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ and let S be the $n \times n$ matrix whose j'th column is $\mathbf{v}_j$, for $j = 1, \ldots, n$. With these notations,

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Proof. By the definition of eigenvectors (and matrix multiplication),

$$AS = A \cdot \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \ldots & \lambda_n\mathbf{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = SD.$$

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So we obtained that

$$AS = SD.$$

Since the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent (see the last slide of Lecture 8), the (column) rank of S is n, which means that S is invertible (as S is an $n \times n$ matrix with determinant rank n).

So using the above equation, we conclude that

$$A = ASS^{-1} = SDS^{-1},$$

as desired. □

Exercise. Calculate $\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}^{100}$, using the diagonalization of $A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$.

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Solution. **1.** Find the eigenvalues of A: (This was an exercise of HW#2.)

$$\begin{vmatrix} 2-x & -3 \\ -1 & 4-x \end{vmatrix} = (2-x)(4-x) - 3 = x^2 - 6x + 5.$$

The roots of $x^2 - 6x + 5$ are $\lambda_1 = 1,\ \lambda_2 = 5$, these are the eigenvalues of A. As the 2×2 matrix A has 2 distinct eigenvalues, A is diagonalizable by the previous theorem.

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2. We have to find an eigenvector $\mathbf{v}_1$ corresponding to the eigenvalue $\lambda_1 = 1$, and an eigenvector $\mathbf{v}_2$ corresponding to the eigenvalue $\lambda_2 = 5$. We saw in Lecture 8, that the eigenvectors corresponding to λ are precisely the non-zerovector solutions of the homogeneous linear system with matrix $A - \lambda I_2$. We we have to pick one such solution for each eigenvalue λ.

For $\lambda_1 = 1$, the matrix $A - \lambda_1 I_2 = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$, so to find an eigenvector $\mathbf{v}_1$, we have to solve the homogeneus linear system

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$$\left[\begin{array}{cc|c} 1^* & -3 & 0 \\ -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The second matrix is in row-echelon form, hence $x_2 = a$ is the free variable and the basic variable is $x_1 = 3x_2$ (from the 1st row), so the solutions are $[3a, a]^T$ where a is arbitrary. To obtain a suitable $\mathbf{v}_1$, pick a non-zerovector solution, e.g., by setting $a = 1$ we get $\mathbf{v}_1 = [3, 1]^T$.

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For $\lambda_2 = 5$, we proceed analogously. Here $A - \lambda_2 I_2 = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix}$.

After solving the linear system $\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -1 & -1 & 0 \end{array}\right]$, we obtain $\mathbf{v}_2 = [-1, 1]^T$ (for example).

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3. Now we apply the previous Theorem with $\lambda_1 = 1$, $\lambda_2 = 5$ and $\mathbf{v}_1 = [3,1]^T, \mathbf{v}_2 = [-1,1]^T$:
With

$$D = \text{Diag}(1,5) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \boxed{\mathbf{v}_1} & \boxed{\mathbf{v}_2} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix},$$

we have that

$$A = SDS^{-1}.$$

Thus by Proposition 2, we know that

$$A^{100} = SD^{100}S^{-1}. \qquad (*)$$

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Thus by Proposition 2, we know that

$$A^{100} = SD^{100}S^{-1}. \tag{$*$}$$

Proposition 1 shows that $D^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix}$. We also need to compute the inverse of S: $S^{-1} = \begin{bmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{bmatrix}$ (see Lec. 9, the details are skipped). Substituting these into ($*$) yields

$$A^{100} = SD^{100}S^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \cdot \begin{bmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(3 + 5^{100}) & \frac{1}{4}(3 - 3 \cdot 5^{100}) \\ \frac{1}{4}(1 - 5^{100}) & \frac{1}{4}(1 + 3 \cdot 5^{100}) \end{bmatrix}.$$

□