Introduction to linear programming

Linear algebra

Lectures 11-12

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1. Initially, the linear system is encoded in an (augmented) basis table as follows.

$$
\begin{cases}
2x_1 - 3x_2 + 5x_3 + x_4 = 1 \\
x_1 - x_2 + 4x_3 - 2x_4 = 2 \\
5x_1 + 2x_3 + 7x_4 + 3x_4 = 6
\end{cases}
\longleftrightarrow
$$

	x_1	x_2	x_3	x_4	
e_1	2	-3	5	1	1
e_2	1	-1	4	-2	2
e_3	5	2	7	3	6

This table is essentially the same as the augmented matrix of the linear system, just the rows
are labeled by e_1, e_2, e_3 (e_i represents the i'th equation here), and the columns labeled by the
variables x_1, x_2, x_3, and the last column has no label. The second vertical bar represents the
equation signs.

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2. By performing elementary basis transformations on this table, we ""move some of the variables into the basis"", and we delete the columns of these variables. For example, we reach to this table:

$$
\left\{
\begin{array}{r}
x_4 + 19x_2 - 11x_3 = 7 \\
48x_2 - 26x_3 = 19 \\
x_1 - 11x_2 + 8x_3 = -3
\end{array}
\right.
\quad \longleftrightarrow \quad
$$

	x_2	x_3	
x_4	19	-11	7
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 & x_2 & x_3 & \\
\hline
x_4 & 19 & -11 & 7 \\
e_2 & 48 & -26 & 19 \\
x_1 & -11 & 8 & -3
\end{array}
$$

It is crucial to understand how the corresponding linear system is reconstructed:

- If a row has label e_i for some i, then the left-hand side of the corresponding equation is the linear expression of the variables in the column labels, with the coefficients contained in this row (between the two vertical bars); and the right-hand side is the last element of this row.

- If a row has label x_i for some i, then corresponding linear equation is obtained in the same way as above, with the modification that the row label x_i is also added to the left-hand side as a new term.

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2. Repeat the following step on the basis table as long as possible: Pick a non-zero element from a row with label e_i (for some i), between the two vertical bar. Perform an elementary basis transformation indicated by the pivot element, and delete the column of the pivot from the new table.

	x_1	x_2	x_3	
x_4	2	-3	5	1
e_2	5	-7	14	4
e_3	-1^*	11	-8	3

$\longrightarrow$

	x_2	x_3	
x_4	19	-11	7
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We ignore the second vertical bar when performing the EBT. The last column is exceptional in one aspect only: The pivot element must not be chosen from that column.

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3. The rows with label e_i (for some i) containing only 0's correspond to the equation $0 = 0$, so these rows can be removed from the obtained table.

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4. Read off the solutions of the linear system. (This will be detailed later, trough examples.)

Key observation. When performing an elementary basis transformation on a basis table corresponding to a linear system, then the obtained basis table corresponds to a linear system which has exactly the same solutions as the original system.

$$
\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
x_4 & 2 & -3 & 5 & 1 \\
e_2 & 5 & -7 & 14 & 4 \\
e_3 & -1^* & 11 & -8 & 3
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c|cc|c}
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\hline
x_4 & 19 & -11 & 7 \\
e_2 & 48 & -26 & 19 \\
x_1 & -11 & 8 & -3
\end{array}
$$

$$
\begin{cases}
x_4 + 2x_1 - 3x_2 + 5x_3 = 1 \\
5x_1 - 7x_2 + 14x_3 = 4 \\
-x_1 + 11x_2 - 8x_3 = 3
\end{cases}
\quad \sim \quad
\begin{cases}
x_4 + 19x_2 - 11x_3 = 7 \\
48x_2 - 26x_3 = 19 \\
x_1 - 11x_2 + 8x_3 = -3
\end{cases}
$$

This is because the effect of the elementary basis transformation on the linear system is that
- the equation of the pivot is divided by the pivot element, so the coefficient indicated by the pivot becomes 1;
- the rectangle rule modifies the other coefficients (and constants) as if we eliminated the variable indicated by the pivot from all other equations, using row operations.

Example 1. Solve the linear system

$$\begin{cases} 4x_1 + 4x_2 + 5x_3 = 6 \\ x_1 + x_2 + 2x_3 = 3 \\ 7x_1 + 7x_2 + 8x_3 = 10. \end{cases}$$

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Solution.

	x_1	x_2	x_3	
e_1	4	4	5	6
e_2	1^*	1	2	3
e_3	7	7	8	10

$\rightarrow$

	x_2	x_3	
e_1	0	-3^*	-6
x_1	1	2	3
e_3	0	-6	-11

$\rightarrow$

	x_2	
x_3	0	2
x_1	1	-1
e_3	0	1

Observe that the 3rd row in the final table corresponds to the equation $0 = 1$, which is never satisfied. So this linear system has no solutions. $\qquad\square$

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Solution.

	x_1	x_2	x_3	
e_1	4	4	5	6
e_2	1^*	1	2	3
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$\rightarrow$

	x_2	x_3	
e_1	0	-3^*	-6
x_1	1	2	3
e_3	0	-6	-11

$\rightarrow$

	x_2	
x_3	0	2
x_1	1	-1
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Observe that the 3rd row in the final table corresponds to the equation $0 = 1$, which is never satisfied. So this linear system has no solutions. $\qquad\square$

Definition. We say that a row of an augmented basis table is contradicting, if it has label e_i (for some i), its last element is non-zero, and its all other elements are 0's.

$$e_i \;\Big|\; 0 \;\; 0 \;\ldots\; 0 \;\; 0 \;\Big|\; c \qquad\qquad (c \neq 0)$$

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Solution.

	x_1	x_2	x_3	
e_1	4	4	5	6
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$\rightarrow$

	x_2	x_3	
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x_1	1	2	3
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$\rightarrow$

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Definition. We say that a row of an augmented basis table is contradicting, if it has label e_i (for some i), its last element is non-zero, and its all other elements are 0's.

Case 1. If a contradicting row appears during the solution of a linear system, then the system has no solutions.

Another example. Solve the linear system

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_2 - x_3 = -2 \\ 4x_1 + x_2 - 3x_3 = 5. \end{cases}$$
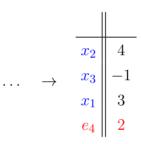
Another example. Solve the linear system

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Solution.

	x_1	x_2	x_3	
e_1	1^*	1	-1	0
e_2	2	-1	-1	-2
e_3	4	1	-3	5

$\rightarrow$

	x_2	x_3	
x_1	1	-1	0
e_2	-3	1	-2
e_3	-3	1^*	5

$\rightarrow$

	x_2	
x_1	-2	5
e_2	0	-7
x_3	-3	5

The 2rd row in the final is contradicting, so this linear system has no solutions. □

Remark. We note that it can happen that a contradicting row has no zeros! This is the case when all variables can be moved into the basis (i.e. there is no column between the two vertical bars), but the table still has a row with label e_i with non-zero (last) element:

$$
\ldots \quad \rightarrow \quad
\begin{array}{c||c}
 & \\
\hline
x_2 & 4 \\
x_3 & -1 \\
x_1 & 3 \\
e_4 & 2 \\
\end{array}
$$

Example 2. Solve the linear system

$$\begin{cases} 2x_1 - x_2 - x_3 = 6 \\ 3x_1 + 4x_2 - 2x_3 = 3 \\ 3x_1 - 2x_2 + 4x_3 = 15. \end{cases}$$

Example 2. Solve the linear system

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Solution.

	x_1	x_2	x_3	
e_1	2	-1^*	-1	6
e_2	3	4	-2	3
e_3	3	-2	4	15

$\rightarrow$

	x_1	x_3	
x_2	-2	1	-6
e_2	11	-6	27
e_3	-1^*	6	3

$\rightarrow$

	x_3	
x_2	-11	-12
e_2	60^*	60
x_1	-6	-3

$\rightarrow$

x_2		-1
x_3		1
x_1		3

The last table yields the solution immediately, because it corresponds to the equations $x_2 = -1$, $x_3 = 1$, $x_1 = 3$. So the system has one solution: $[3, -1, 1]^T$ in vector form. (Remember to write the values of the variables to the vector form in correct order.)   □

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Solution.

	x_1	x_2	x_3	
e_1	2	-1^*	-1	6
e_2	3	4	-2	3
e_3	3	-2	4	15

$\rightarrow$

	x_1	x_3	
x_2	-2	1	-6
e_2	11	-6	27
e_3	-1^*	6	3

$\rightarrow$

	x_3	
x_2	-11	-12
e_2	60^*	60
x_1	-6	-3

$\rightarrow$

x_2	-1
x_3	1
x_1	3

The last table yields the solution immediately, because it corresponds to the equations $x_2 = -1$, $x_3 = 1$, $x_1 = 3$. So the system has one solution: $[3, -1, 1]^T$ in vector form. (Remember to write the values of the variables to the vector form in correct order.)　□

Case 2. If all variables can be moved into the basis and there are no contradicting rows in the final table, then the linear system has exactly one solution: The values of the variables are provided by the corresponding rows of the final table.

Another example. Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = 1 \\ 2x_1 - x_2 + 4x_3 = 2 \\ 4x_1 + 4x_2 + 3x_3 = 1. \end{cases}$$

Another example. Solve the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = 1 \\ 2x_1 - x_2 + 4x_3 = 2 \\ 4x_1 + 4x_2 + 3x_3 = 1. \end{cases}$$

Solution.

	x_1	x_2	x_3	
e_1	1^*	1	1	1
e_2	2	3	1	1
e_3	2	-1	4	2
e_4	4	4	3	1

$\rightarrow$

	x_2	x_3	
x_1	1	1	1
e_2	1	-1	-1
e_3	-3	2	0
e_4	0	-1^*	-3

$\rightarrow$

	x_2	
x_1	1	-2
e_2	1^*	2
e_3	-3	-6
x_3	0	3

$\rightarrow$

x_1	-4
x_2	2
e_3	0
x_3	3

All variables could be moved to the basis, and there are no contradicting rows in the last table (the 3rd row could be removed, because it is an all-zero row with label e_i, and hence it corresponds to the equation $0 = 0$). So the linear system has one solution, which can be read off from the rows of the final table: $x_1 = -4$, $x_2 = 2$, $x_3 = 3$, or $[-4, 2, 3]^T$ in vector form. $\square$

Example 3. Solve the linear system

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2 \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3 \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 9 \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1. \end{cases}$$

Example 3. Solve the linear system

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2 \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3 \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 9 \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1. \end{cases}$$

Solution.

	x_1	x_2	x_3	x_4	x_5	
e_1	2^*	-1	1	2	3	2
e_2	6	-3	2	4	5	3
e_3	6	-3	4	8	13	9
e_4	4	-2	1	1	2	1

$\rightarrow$

	x_2	x_3	x_4	x_5	
x_1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	1
e_2	0	-1	-2	-4	-3
e_3	0	1^*	2	4	3
e_4	0	-1	-3	-4	-3

$\rightarrow$

	x_2	x_4	x_5	
x_1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0	0
x_3	0	2	4	3
e_4	0	-1^*	0	0

$\rightarrow$

	x_2	x_5	
x_1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0
x_3	0	4	3
x_4	0	0	0

(The 2nd row can be discarded, because it is an all-zero row with label e_i.) There are no contradicting rows, but not all variables could be moved into the basis.

In this case, the remaining variables in column labels are the free variables: x_2, x_5; and the variables in the final basis (in row labels) are the basic variables: x_1, x_3, x_4.

$$\vdots$$

	x_1	x_2	x_3	x_4	x_5	
e_1	2^*	-1	1	2	3	2
e_2	6	-3	2	4	5	3
e_3	6	-3	4	8	13	9
e_4	4	-2	1	1	2	1

$\rightarrow$

	x_2	x_3	x_4	x_5	
x_1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	1
e_2	0	-1	-2	-4	-3
e_3	0	1^*	2	4	3
e_4	0	-1	-3	-4	-3

$\rightarrow$

	x_2	x_4	x_5	
x_1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0	0
x_3	0	2	4	3
e_4	0	-1^*	0	0

$\rightarrow$

	x_2	x_5	
x_1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0
x_3	0	4	3
x_4	0	0	0

Free variables: x_2, x_5. Basic variables: x_1, x_3, x_4.

The expression of basic variables in terms of free variables can be immediately read off from the corresponding rows of the final table:

$$x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_5 = -\frac{1}{2} \quad \implies \quad \boxed{x_1 = -\frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_5}$$

$$x_3 + 4x_5 = 3 \quad \implies \quad \boxed{x_3 = 3 - 4x_5}$$

$$x_4 = 0 \quad \implies \quad \boxed{x_4 = 0}$$

The free variables can take any value: $x_2 = a$, $x_5 = b$; so the linear system has infinitely many solutions: $[-\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b,\ a,\ 3 - 4b,\ 0,\ b]^T$, where a and b can be arbitrary numbers.　　　$\square$

	x_1	x_2	x_3	x_4	x_5	
e_1	2^*	-1	1	2	3	2
e_2	6	-3	2	4	5	3
e_3	6	-3	4	8	13	9
e_4	4	-2	1	1	2	1

$\rightarrow$

	x_2	x_3	x_4	x_5	
x_1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	1
e_2	0	-1	-2	-4	-3
e_3	0	1^*	2	4	3
e_4	0	-1	-3	-4	-3

$\rightarrow$

	x_2	x_4	x_5	
x_1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0	0
x_3	0	2	4	3
e_4	0	-1^*	0	0

$\rightarrow$

	x_2	x_5	
x_1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
e_2	0	0	0
x_3	0	4	3
x_4	0	0	0

Case 3. If not all variables can be moved into the basis and there are no contradicting rows in the final table, then the linear system has infinitely many solutions: The column labels of the final table are the free variables, and the rest of the variables (in the row labels) are the basic variables. The basic variables can be directly expressed in terms of free variables, using the corresponding rows of the final table (see the solution of Example 3).

Let's turn our attention to linear optimization:

Definition. A standard linear programming (LP) problem is the following:

Minimize the linear objective function $z = c_1 x_1 + \cdots + c_n x_n$ of the variables $x_1, \ldots, x_n$ subject to a finite set of linear constraints:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1 \\ \qquad\qquad\qquad\qquad\qquad\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m, \end{cases}$$

where the c_i's and $a_{i,j}$'s are arbitrary given numbers, and the b_i's are given nonnegative numbers, and the variables are all required to be nonnegative.

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where the c_i's and $a_{i,j}$'s are arbitrary given numbers, and the b_i's are given nonnegative numbers, and the variables are all required to be nonnegative.

Remark 1. More precisely, the objective function should be written as

$$z(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n,$$

since z is a function of $x_1, \ldots, x_n$.

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where the c_i's and $a_{i,j}$'s are arbitrary given numbers, and the b_i's are given nonnegative numbers, and the variables are all required to be nonnegative.

Remark 1. More precisely, the objective function should be written as

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since z is a function of $x_1, \ldots, x_n$.

Remark 2. If $\mathcal{S}$ denotes the set of those n-tuples of nonnegative numbers $[x_1, \ldots, x_n]^T$ that satisfy all m linear inequalities above, then the above LP problem is to minimize the function z on the set $\mathcal{S}$. (And we also want to find an n-tuple $\mathbf{s} \in \mathcal{S}$ where the minimum is attained.)

Definition. A standard linear programming (LP) problem is the following:

Minimize the linear objective function $z = c_1 x_1 + \cdots + c_n x_n$ of the variables $x_1, \ldots, x_n$ subject to a finite set of linear constraints:

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n \leq b_1 \\ \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n \leq b_m, \end{cases}$$

where the c_i's and $a_{i,j}$'s are arbitrary given numbers, and the b_i's are given nonnegative numbers, and the variables are all required to be nonnegative.

Remark 3. In short, we will write the above standard LP problem as

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n \leq b_1 \\ \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n \leq b_m \\ \qquad\qquad\qquad (x_1, x_2, \ldots, x_n \geq 0) \end{cases}$$

$$c_1 x_1 + \cdots + c_n x_n \to \min$$

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where the c_i's and $a_{i,j}$'s are arbitrary given numbers, and the b_i's are given nonnegative numbers, and the variables are all required to be nonnegative.

Example of a standard LP problem.

$$\begin{cases} 3x_1 + 4x_2 + x_3 \leq 56 \\ x_1 - 5x_2 + 2x_3 \leq 10 \\ x_1 + x_2 - x_3 \leq 20 \\ (x_1, x_2, x_3 \geq 0) \end{cases}$$
$$-2x_1 + 3x_2 - x_3 \to \min$$

We will see that standard LP problems can be solved using the so-called simplex algorithm. However, standard LP problems have a very special form.

Sometimes we have to maximize the objective function z instead of minimization, some of the constraints may be linear equalities, some of the constraints have negative constant on their right-hand side, and some of the variables might be allowed to be negative. Optimization problems of this form are called general LP problems.

Example of a general LP problem.

$$\begin{cases} 3x_1 + 4x_2 + x_3 \leq 56 \\ x_1 - 5x_2 + 2x_3 \leq 17 \\ x_1 + x_2 - x_3 = 20 \\ (x_1, x_3 \geq 0) \end{cases}$$
$$4x_1 + 3x_2 - x_3 \rightarrow \max$$

We will see that standard LP problems can be solved using the so-called simplex algorithm. However, standard LP problems have a very special form.

Sometimes we have to maximize the objective function z instead of minimization, some of the constraints may be linear equalities, some of the constraints have negative constant on their right-hand side, and some of the variables might be allowed to be negative. Optimization problems of this form are called general LP problems.

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$$4x_1 + 3x_2 - x_3 \rightarrow \max$$

Remark. It turns out that every general LP problem can be reduced to a standard LP problem (and so can be solved by simplex algorithm). That is why we only focus on standard LP problems. We only give an insight through examples how this reduction is done.

Online (general) LP solver: https://linprog.com/

Example 1. A company produces two products: chairs and tables. They make a profit of $40 on each chair and $50 on each table. A chair requres the following resources to produce: 2 man-hours, 3 hours of machine time, and 1 unit of wood. The table requires 2 man-hours, 1 hour of machine time, and 4 units of wood. The factory has 60 man-hours, 75 machine hours, and 84 units of wood available each day. How should the resources (man-hours, machine-hours, and wood) be allocated between the two products in order to maximize the factory's profit?

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The corresponding LP problem.

$$\begin{cases} 2c + 2t \le 60 \\ 3c + t \le 75 \\ c + 4t \le 84 \\ \quad (c, t \ge 0) \end{cases}$$

$$40c + 50t \to \max$$

Example 2. A feed company wants each feed bag that they produce to contain a minimum of 120 units of protein and 80 units of calcium. Corn contains 10 units of protein and 5 units of calcium per pound, and bone-meal contains 2 units of protein and 5 units of calcium per pound. If corn costs 8 cents per pound and bone-meal costs 4 cents per pound, how much of each should they put in each bag, in order to minimize costs?

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The corresponding LP problem.

$$\begin{cases} 10c + 2b \geq 120 \\ 5c + 5b \geq 80 \\ (c, b \geq 0) \end{cases}$$

$$8c + 4b \rightarrow \min$$

As noted before, general LP problems can be reduced to standard LP problems. We demonstrate this by showing how to reduce the maximization of z to minimization. (This is the most common reduction needed in practice.)

Claim. If an LP problem $\mathcal{P}$ asks for maximization of the objective function z then it is enough to solve the LP problem $\mathcal{P}'$ that asks for minimization of the objective function $-z$ subject to the same constraints:

If the minimum of $-z$ is M, then the maximum of z is $-M$ (attained at the same n-tuple). Or if $-z$ is unbounded below, then z is unbounded above.

Example.

$$\begin{cases} 3x_1 + 4x_2 + x_3 \leq 56 \\ x_1 - 5x_2 + 2x_3 \leq 10 \\ x_1 + x_2 - x_3 \leq 20 \\ (x_1, x_2, x_3 \geq 0) \end{cases} \qquad \equiv \qquad \begin{cases} 3x_1 + 4x_2 + x_3 \leq 56 \\ x_1 - 5x_2 + 2x_3 \leq 10 \\ x_1 + x_2 - x_3 \leq 20 \\ (x_1, x_2, x_3 \geq 0) \end{cases}$$

$$2x_1 - 3x_2 + x_3 \to \max \qquad\qquad\qquad -2x_1 + 3x_2 - x_3 \to \min$$

Now we present the simplex algorithm that can be used to solve standard LP problems:

Exercise. Solve the following standard LP problem:

$$\begin{cases} 3x_1 + x_2 + x_3 \le 60 \\ x_1 - x_2 + 2x_3 \le 10 \\ x_1 + x_2 - x_3 \le 20 \\ (x_1, x_2, x_3 \ge 0) \end{cases}$$

$$-2x_1 + x_2 - x_3 \to \min$$

Step 1. Create the (initial) simplex tableau of the standard LP problem:

$$\begin{cases} 3x_1 + x_2 + x_3 \leq 60 \\ x_1 - x_2 + 2x_3 \leq 10 \\ x_1 + x_2 - x_3 \leq 20 \\ (x_1, x_2, x_3 \geq 0) \end{cases}$$

$$-2x_1 + x_2 - x_3 \to \min$$

$\rightsquigarrow$

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

- The simplex tableau is similar to the augmented basis table of linear equations, just with one extra (unlabeled) row added, separated by a horizontal bar.
- The column labels are the variables of the LP problem. In the row labels new distinct variables are introduced (these "virtual" variables are called slack variables).
- The linear constraints (inequalities) are encoded in the labeled rows of the table, in a natural way, see the above example. (The inequalities $x_1, \ldots, x_n \geq 0$ are discarded here.)
- The last (unlabeled) row of the table contains the coefficients of the objective function, between the two vertical bars.
- The bottom-right element is always 0 initially.

Step 2/a. Find the smallest negative number (coefficient) in the last row of the tableau between the two vertical bars. (If there are no negative numbers there, we are done, go to Step 4.) In case of a tie, choose arbitrarily.

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

Step 2/a. Find the smallest negative number (coefficient) in the last row of the tableau between the two vertical bars. (If there are no negative numbers there, we are done, go to Step 4.) In case of a tie, choose arbitrarily.

Step 2/b. Consider the column of the found negative coefficient. For each positive number a in this column, calculate the ratio b/a, where b is the last element in the row of a. (If there are no positive numbers in that column, we are done, go to Step 4.) Find the smallest b/a ratio among these, and designate the corresponding a element as pivot. In case of a tie, choose arbitrarily.

	x_1	x_2	x_3		ratio
x_4	3	1	1	60	$60/3 = 20$
x_5	1^*	-1	2	10	$10/1 = \boxed{10}$
x_6	1	1	-1	20	$20/1 = 20$
	-2	1	-1	0	

Step 2/a. Find the smallest negative number (coefficient) in the last row of the tableau between the two vertical bars. (If there are no negative numbers there, we are done, go to Step 4.) In case of a tie, choose arbitrarily.

Step 2/b. Consider the column of the found negative coefficient. For each positive number a in this column, calculate the ratio b/a, where b is the last element in the row of a. (If there are no positive numbers in that column, we are done, go to Step 4.) Find the smallest b/a ratio among these, and designate the corresponding a element as pivot. In case of a tie, choose arbitrarily.

Step 3. Perform an elementary basis transformation on the simplex tableau, indicated by the selected pivot element. (All numbers of the tableaux are considered, when performing the EBT.) In this algorithm the column of the pivot must not be deleted!

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1^*	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

$\rightarrow$

	x_5	x_2	x_3	
x_4	-3	4	-5	30
x_1	1	-1	2	10
x_6	-1	2	-3	10
	2	-1	3	20

Step 4. Repeat Steps 2-3 (pivot selection, then EBT) until one of the following occurs:

Case 1: If there are no negative coefficients in the last row of the tableaux, then the minimum of the objective function z subject to the constraints is $-c$, where c denotes the last element of the last row. And this minimum value is attained when the non-slack variables in row labels are set to the last element of their row, and the non-slack variables in column labels are set to 0. (As you see, we simply ignore the slack variables here.)

Case 2: If there exists a negative coefficient in the last row of the tableaux whose column does not contain any positive number, then the objective function z subject to the constraints is unbounded below (i.e., for any real number r, the function z attains smaller values than r on the solution set of the constraints), there is no minimum.

$$
\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
x_4 & 3 & 1 & 1 & 60 \\
x_5 & 1^* & -1 & 2 & 10 \\
x_6 & 1 & 1 & -1 & 20 \\
\hline
 & -2 & 1 & -1 & 0
\end{array}
\qquad \rightarrow \qquad
\begin{array}{c|ccc|c}
 & x_5 & x_2 & x_3 & \\
\hline
x_4 & -3 & 4 & -5 & 30 \\
x_1 & 1 & -1 & 2 & 10 \\
x_6 & -1 & 2 & -3 & 10 \\
\hline
 & 2 & -1 & 3 & 20
\end{array}
$$

Step 4. Repeat Steps 2-3 (pivot selection, then EBT) until one of the following occurs:

Case 1: If there are no negative coefficients in the last row of the tableaux, then the minimum of the objective function z subject to the constraints is $-c$, where c denotes the last element of the last row. And this minimum value is attained when the non-slack variables in row labels are set to the last element of their row, and the non-slack variables in column labels are set to 0. (As you see, we simply ignore the slack variables here.)

Case 2: If there exists a negative coefficient in the last row of the tableaux whose column does not contain any positive number, then the objective function z subject to the constraints is unbounded below (i.e., for any real number r, the function z attains smaller values than r on the solution set of the constraints), there is no minimum.

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1^*	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

$\rightarrow$

	x_5	x_2	x_3		ratio
x_4	-3	4	-5	30	$30/4 = 7.5$
x_1	1	-1	2	10	
x_6	-1	2^*	-3	10	$10/2 = \boxed{5}$
	2	-1	3	20	

Step 4. Repeat Steps 2-3 (pivot selection, then EBT) until one of the following occurs:

Case 1: If there are no negative coefficients in the last row of the tableaux, then the minimum of the objective function z subject to the constraints is $-c$, where c denotes the last element of the last row. And this minimum value is attained when the non-slack variables in row labels are set to the last element of their row, and the non-slack variables in column labels are set to 0. (As you see, we simply ignore the slack variables here.)

Case 2: If there exists a negative coefficient in the last row of the tableaux whose column does not contain any positive number, then the objective function z subject to the constraints is unbounded below (i.e., for any real number r, the function z attains smaller values than r on the solution set of the constraints), there is no minimum.

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1^*	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

$\rightarrow$

	x_5	x_2	x_3	
x_4	-3	4	-5	30
x_1	1	-1	2	10
x_6	-1	2^*	-3	10
	2	-1	3	20

$\rightarrow$

	x_5	x_6	x_3	
x_4	-1	-2	1	10
x_1	$1/2$	$1/2$	$1/2$	15
x_2	$-1/2$	$1/2$	$-3/2$	5
	$3/2$	$1/2$	$3/2$	25

Step 4. Repeat Steps 2-3 (pivot selection, then EBT) until one of the following occurs:

Case 1: If there are no negative coefficients in the last row of the tableaux, then the minimum of the objective function z subject to the constraints is $-c$, where c denotes the last element of the last row. And this minimum value is attained when the non-slack variables in row labels are set to the last element of their row, and the non-slack variables in column labels are set to 0. (As you see, we simply ignore the slack variables here.)

Case 2: ...

Solution. The minimum of the objective function (subject to the constraints) is -25. This value is attained at $x_1 = 15$, $x_2 = 5$, $x_3 = 0$.   □

	x_1	x_2	x_3	
x_4	3	1	1	60
x_5	1^*	-1	2	10
x_6	1	1	-1	20
	-2	1	-1	0

$\rightarrow$

	x_5	x_2	x_3	
x_4	-3	4	-5	30
x_1	1	-1	2	10
x_6	-1	2^*	-3	10
	2	-1	3	20

$\rightarrow$

	x_5	x_6	x_3	
x_4	-1	-2	1	10
x_1	$1/2$	$1/2$	$1/2$	15
x_2	$-1/2$	$1/2$	$-3/2$	5
	$3/2$	$1/2$	$3/2$	25

Remark 1. The ratio comparison in Step 2/b ensures that during the algorithm the elements in the last column of the simplex tableau are all nonnegative, always. If this is not the case, then something went wrong ...

Remark 2. Unfortunately, the algorithm can lead to cycling (that is, Step 4 is never reached, because the sequence of simplex tableaus we creating contain the same tableau twice and so continue to loop indefinitely). For example, see Exercise 5.3.7 in the lecture notes. This can be eliminated by modifying the column selection in Step 2/a. We skip the details.

Remark 3. Keep in mind that the simplex algorithm can be used to solve standard LP problems only. General LP problems must be reduced to a standard LP problem first, as illustrated on slide 'The reduction of maximization to minimization'.

Exercise 2. Solve the following standard LP problem:

$$\begin{cases} -x_1 + x_2 \leq 1 \\ x_1 - 2x_2 \leq 2 \\ (x_1, x_2 \geq 0) \end{cases}$$

$$-2x_1 - x_2 \to \min$$

Exercise 2. Solve the following standard LP problem:

$$\begin{cases} -x_1 + x_2 \le 1 \\ x_1 - 2x_2 \le 2 \\ \quad (x_1, x_2 \ge 0) \end{cases}$$

$$-2x_1 - x_2 \to \min$$

Solution.

	x_1	x_2	
x_3	-1	1	-1
x_4	1^*	-2	2
	-2	-1	0

$\rightarrow$

	x_4	x_2	
x_3	1	-1	1
x_2	1	-2	2
	2	-5	4

There exists a negative coefficient in the last row of the tableaux whose column does not contain any positive number, so the objective function z subject to the constraints is unbounded below, there is no minimum. (See Case 2 in Step 4.)   □

Worked-out example 1.   CLICK HERE

Worked-out example 2.   CLICK HERE

Worked-out example 3.   See Example 5.10 in the lecture notes. (Fractions are involved here.)