Leontief input/output model

Linear algebra

Lecture 10

Read also: Chapter 6 in the lecture notes

(See the Documents folder in CooSpace)

Gábor V. Nagy

Bolyai Intitute

Szeged, 2020.

Assume that an economy consists of n interdependent sectors (or industries). Each sector will consume some of the goods produced by the other sectors, including itself (for example, a power-generating plant uses some of its own power for production). We assume that the economy is closed, which means that it satisfies its own needs: no goods leave or enter the system.

Assume that an economy consists of n interdependent sectors (or industries). Each sector will consume some of the goods produced by the other sectors, including itself (for example, a power-generating plant uses some of its own power for production). We assume that the economy is closed, which means that it satisfies its own needs: no goods leave or enter the system.

Definition. The input-output matrix (for n sectors) in an $n \times n$ matrix in which the j'th column contains the inputs needed from other sectors to produce one unit of output (where the i'th element corresponds to the i'th sector).

Example. Assume that the economy has 3 sectors, Agriculture, Construction and Transportation, with input-output matrix

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

E.g., the 2nd column shows that for producing 1 unit in Construction, it is needed 0.3 unit of 'products' from Agriculture, 0.1 unit of products from Construction, and 0.2 unit of products from Transportation as input.

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational ‘products’. How many units of products does the economy need to produce in total?

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \quad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational ‘products’. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- We produce the required 2, 1, and 2 units of agricultural, construction, and transportational products.

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- We produce the required 2, 1, and 2 units of agricultural, construction, and transportational products.
- For producing 2 units of agricultural products, we also need 0.6 unit of agricultural, 0.4 unit of construction and 0 unit of transportational products. For producing 1 unit of construction products, we also need 0.3 unit of agricultural, 0.1 unit of construction and 0.2 unit of transportational products. For producing 2 units of transportational products, we also need 0.4 unit of agricultural, 0.4 unit of construction and 0.4 unit of transportational products.

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- We produce the required 2, 1, and 2 units of agricultural, construction, and transportational products.
- For producing 2 units of agricultural products, we also need 0.6 unit of agricultural, 0.4 unit of construction and 0 unit of transportational products. For producing 1 unit of construction products, we also need 0.3 unit of agricultural, 0.1 unit of construction and 0.2 unit of transportational products. For producing 2 units of transportational products, we also need 0.4 unit of agricultural, 0.4 unit of construction and 0.4 unit of transportational products.
- So we additionally need $0.6 + 0.3 + 0.4 = 1.3$ units of agricultural, $0.4 + 0.1 + 0.4 = 0.9$ unit of construction, and $0 + 0.2 + 0.4 = 0.6$ unit of transportational products. BUT ...

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \quad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- BUT we also need inputs for producing these additional products (1.3 units of agricultural, 0.9 unit of construction, 0.6 transportational): For producing 1.3 units of agricultural products, we need more $1.3 \cdot 0.3$ unit of agricultural, $1.3 \cdot 0.2$ unit of construction and 0 unit of transportational products, and so on, we can calculate the required new input in the same way as above.

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \quad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational ‘products’. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- BUT we also need inputs for producing these additional products (1.3 units of agricultural, 0.9 unit of construction, 0.6 transportational): For producing 1.3 units of agricultural products, we need more $1.3 \cdot 0.3$ unit of agricultural, $1.3 \cdot 0.2$ unit of construction and 0 unit of transportational products, and so on, we can calculate the required new input in the same way as above.
- And we should repeat this input calculation for the obtained extra products, and so on, and so on, this process never ends . . .

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

First try. It seems easy at first sight:

- BUT we also need inputs for producing these additional products (1.3 units of agricultural, 0.9 unit of construction, 0.6 transportational): For producing 1.3 units of agricultural products, we need more $1.3 \cdot 0.3$ unit of agricultural, $1.3 \cdot 0.2$ unit of construction and 0 unit of transportational products, and so on, we can calculate the required new input in the same way as above.
- And we should repeat this input calculation for the obtained extra products, and so on, and so on, this process never ends ...

But we can still solve this problem.

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Question. The order is to produce 2, 1, and 2 units of agricultural, construction, and transportational 'products'. How many units of products does the economy need to produce in total?

Some notations. We collect the demands into the final demand vector $\mathbf{d} = [2, 1, 2]^T$.

Let x_1, x_2 and x_3 denote the total number of units of products that is required to produce in Agriculute, Construction, and Transportation (respectively) to fulfill the demands in $\mathbf{d}$. The question asks to determine the total output vector $\mathbf{x} = [x_1, x_2, x_3]^T$.

Some notations. We collect the demands into the final demand vector $\mathbf{d} = [2, 1, 2]^T$.

Let x_1, x_2 and x_3 denote the total number of units of products that is required to produce in Agriculute, Construction, and Transportation (respectively) to fulfill the demands in $\mathbf{d}$. The question asks to determine the total output vector $\mathbf{x} = [x_1, x_2, x_3]^T$.

Some notations. We collect the demands into the final demand vector $\mathbf{d} = [2, 1, 2]^T$.

Let x_1, x_2 and x_3 denote the total number of units of products that is required to produce in Agriculute, Construction, and Transportation (respectively) to fulfill the demands in $\mathbf{d}$. The question asks to determine the total output vector $\mathbf{x} = [x_1, x_2, x_3]^T$.

Solution. The crucial observation is that

$$\mathbf{x} - A\mathbf{x} = \mathbf{d} \tag{1}$$

(where A is the input-output matrix), because $A\mathbf{x}$ is the vector of (total) required inputs for producing $\mathbf{x}$ products (no matter how $\mathbf{x}$ is distributed among the logical levels seen before), as

$$A\mathbf{x} = \begin{bmatrix} 0.3 \cdot x_1 + 0.3 \cdot x_2 + 0.2 \cdot x_3 \\ 0.2 \cdot x_1 + 0.1 \cdot x_2 + 0.2 \cdot x_3 \\ 0 \cdot x_1 + 0.2 \cdot x_2 + 0.2 \cdot x_3 \end{bmatrix};$$

so the vector $\mathbf{x} - A\mathbf{x}$ contains the total number of units of those products that are outputs but not inputs, and this vector should be equal to $\mathbf{d}$. The equation (1) can be rewritted as

$$(I_3 - A)\mathbf{x} = \mathbf{d},$$

where I_3 is the identity matrix of size 3×3. So if the matrix $I - A$ is invertible, we get that

$$\boxed{\mathbf{x} = (I - A)^{-1}\mathbf{d}}.$$

In this example

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

We have to calculate the inverse of

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 & -0.2 \\ -0.2 & 0.9 & -0.2 \\ 0 & -0.2 & 0.8 \end{bmatrix},$$

which is

$$(I - A)^{-1} = \begin{bmatrix} 34/21 & 2/3 & 4/7 \\ 8/21 & 4/3 & 3/7 \\ 2/21 & 1/3 & 19/14 \end{bmatrix} \approx \begin{bmatrix} 1.62 & 0.67 & 0.57 \\ 0.38 & 1.33 & 0.43 \\ 0.10 & 0.33 & 1.36 \end{bmatrix},$$

see Lecture 9. So we obtain the desired total output by matrix multiplication:

$$\mathbf{x} = (I - A)^{-1}\mathbf{d} = [106/21, 62/21, 68/21]^T \approx [5.05, 2.95, 3.24]^T. \qquad \square$$

The previous argument works in general:

Theorem 1. Let A be the input-output matrix of an economy with n sectors, and let $\mathbf{d} \in \mathbb{R}^n$ be a final demand vector. If all entries of A and $\mathbf{d}$ are nonnegative, and the matrix $I_n - A$ is invertible, then the total output vector is

$$\mathbf{x} = (I_n - A)^{-1}\mathbf{d}.$$

We note that the elements of this vector $\mathbf{x}$ are also nonnegative.

The previous argument works in general:

Theorem 1. Let A be the input-output matrix of an economy with n sectors, and let $\mathbf{d} \in \mathbb{R}^n$ be a final demand vector. If all entries of A and $\mathbf{d}$ are nonnegative, and the matrix $I_n - A$ is invertible, then the total output vector is

$$\mathbf{x} = (I_n - A)^{-1}\mathbf{d}.$$

We note that the elements of this vector $\mathbf{x}$ are also nonnegative.

Definition. For an $n \times n$ matrix A, the matrix $(I_n - A)^{-1}$ is called the Leontief inverse of A (if it exists).

The previous argument works in general:

Theorem 1. Let A be the input-output matrix of an economy with n sectors, and let $\mathbf{d} \in \mathbb{R}^n$ be a final demand vector. If all entries of A and $\mathbf{d}$ are nonnegative, and the matrix $I_n - A$ is invertible, then the total output vector is

$$\mathbf{x} = (I_n - A)^{-1}\mathbf{d}.$$

We note that the elements of this vector $\mathbf{x}$ are also nonnegative.

Definition. For an $n \times n$ matrix A, the matrix $(I_n - A)^{-1}$ is called the Leontief inverse of A (if it exists).

Theorem 2. If the Leontief inverse of an input-output matrix contains a negative entry, then the corresponding economy is not productive. Otherwise, if the Leontief inverse exists and it has no negative elements, then the economy is productive.

Note. 'Not productive' means here that the production of a unit of product in a sector would require the production of infinite number of units of products in the total output.

The previous argument works in general:

Theorem 1. Let A be the input-output matrix of an economy with n sectors, and let $\mathbf{d} \in \mathbb{R}^n$ be a final demand vector. If all entries of A and $\mathbf{d}$ are nonnegative, and the matrix $I_n - A$ is invertible, then the total output vector is

$$\mathbf{x} = (I_n - A)^{-1}\mathbf{d}.$$

We note that the elements of this vector $\mathbf{x}$ are also nonnegative.

Definition. For an $n \times n$ matrix A, the matrix $(I_n - A)^{-1}$ is called the Leontief inverse of A (if it exists).

Theorem 2. If the Leontief inverse of an input-output matrix contains a negative entry, then the corresponding economy is not productive. Otherwise, if the Leontief inverse exists and it has no negative elements, then the economy is productive.

Example. The input-output matrix $A = \begin{bmatrix} 1.1 & 0.7 & 0.3 \\ 1.1 & 0.1 & 1.2 \\ 1.2 & 1.2 & 0.1 \end{bmatrix}$ has Leontief inverse $(I_3 - A)^{-1} \approx$ $\begin{bmatrix} 5.07 & -1.03 & -2.87 \\ -4.72 & 0.89 & 3.51 \\ -4.26 & 1.70 & 2.34 \end{bmatrix}$ with negative entries, so the economy is not productive.

In order to determine the Leontief inverse of a matrix, it is necessary to calculate the inverse of a matrix. This can be always done using basis transformation (c.f. Lecture 9).

For 2×2 matrices, the basis transformation yields the following quick formula:

The inverse of a 2×2 matrix. Assume that the 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible (that is, $|M| \neq 0$). Then

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example.

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \cdot 6 - 7 \cdot 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}.$$

Definition. In an economy with n sectors, the price vector $\mathbf{p} = [p_1, \ldots, p_n]$ contains the price of 1 unit of product, for each sector (the i'th element corresponds to the i'th sector).

The price vector is a ROW vector (not column vector, like the final demand and total output vectors).

Definition. In an economy with n sectors, the price vector $\mathbf{p} = [p_1, \ldots, p_n]$ contains the price of 1 unit of product, for each sector (the i'th element corresponds to the i'th sector).

The price vector is a ROW vector (not column vector, like the final demand and total output vectors).

Example. Recall the previous example:

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Assume that the price vector is $\mathbf{p} = [2, 1, 5]$. Find the profit/loss in producing one unit of product for each sectors.

Example. Recall the previous example:

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Assume that the price vector is $\mathbf{p} = [2, 1, 5]$. Find the profit/loss in producing one unit of product for each sectors.

Solution. For producing one unit of product in Agriculture, we need 0.3 unit of agricultural products, 0.2 unit of construction products, and 0 unit of transportational products, which costs $2 \cdot 0.3 + 1 \cdot 0.2 + 5 \cdot 0$. Similarly, the cost of producing one unit of product in Construction is $2 \cdot 0.3 + 1 \cdot 0.1 + 5 \cdot 0.2$, and this cost is $2 \cdot 0.2 + 1 \cdot 0.2 + 5 \cdot 0.2$ in Transportation. Observe that these costs are precisely the entries of the (row) vector $\mathbf{p}A = [0.8, 1.7, 1.6]$.

Example. Recall the previous example:

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Assume that the price vector is $\mathbf{p} = [2, 1, 5]$. Find the profit/loss in producing one unit of product for each sectors.

Solution. For producing one unit of product in Agriculture, we need 0.3 unit of agricultural products, 0.2 unit of construction products, and 0 unit of transportational products, which costs $2 \cdot 0.3 + 1 \cdot 0.2 + 5 \cdot 0$. Similarly, the cost of producing one unit of product in Construction is $2 \cdot 0.3 + 1 \cdot 0.1 + 5 \cdot 0.2$, and this cost is $2 \cdot 0.2 + 1 \cdot 0.2 + 5 \cdot 0.2$ in Transportation. Observe that these costs are precisely the entries of the (row) vector $\mathbf{p}A = [0.8, 1.7, 1.6]$.

This means that in producing one unit of product in Agriculture, the profit is $2 - 0.8 = 1.2$. Similarly, the profit/loss in producing one product is in Costruction $1 - 1.7 = -0.7$, and it is $5 - 1.6 = 3.4$ in Transportation. Observe that the profit/loss vector is $\mathbf{p} - \mathbf{p}A$. $\square$

Example. Recall the previous example:

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix} \qquad \longleftrightarrow$$

	Agr.	Constr.	Transp.
Agr.	0.3	0.3	0.2
Constr.	0.2	0.1	0.2
Transp.	0	0.2	0.2

Assume that the price vector is $\mathbf{p} = [2, 1, 5]$. Find the profit/loss in producing one unit of product for each sectors.

Solution. For producing one unit of product in Agriculture, we need 0.3 unit of agricultural products, 0.2 unit of construction products, and 0 unit of transportational products, which costs $2 \cdot 0.3 + 1 \cdot 0.2 + 5 \cdot 0$. Similarly, the cost of producing one unit of product in Construction is $2 \cdot 0.3 + 1 \cdot 0.1 + 5 \cdot 0.2$, and this cost is $2 \cdot 0.2 + 1 \cdot 0.2 + 5 \cdot 0.2$ in Transportation. Observe that these costs are precisely the entries of the (row) vector $\mathbf{p}A = [0.8, 1.7, 1.6]$.

This means that in producing one unit of product in Agriculture, the profit is $2 - 0.8 = 1.2$. Similarly, the profit/loss in producing one product is in Costruction $1 - 1.7 = -0.7$, and it is $5 - 1.6 = 3.4$ in Transportation. Observe that the profit/loss vector is $\mathbf{p} - \mathbf{p}A$. $\square$

We can conclude that Agriculture and Transportation run at profit, and Construction runs at loss with this price vector.

Theorem 3. Assume that an economy has input-output matrix A. With price vector $\mathbf{p}$, the profit/loss vector (in producing 1 unit of products) is $\mathbf{p} - \mathbf{p}A$.

Theorem 3. Assume that an economy has input-output matrix A. With price vector $\mathbf{p}$, the profit/loss vector (in producing 1 unit of products) is $\mathbf{p} - \mathbf{p}A$.

Theorem 4. An economy is productive (i.e. the Leontief inverse of the input-matrix A contains no negative entries) if and only if there exists a price vector $\mathbf{p}$ with non-negative entries for which each sectors run at profit (i.e. the entries of $\mathbf{p} - \mathbf{p}A$ are all non-negative).

Exercise. An economy consists of 2 sectors with input-output matrix $A = \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

(a) Decide whether this economy is productive or not.

(b) Compute the total output vector $\mathbf{x}$, if the final demand vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(c) Which sectors run at profit, if the price vector is $[4, 3]$?

Exercise. An economy consists of 2 sectors with input-output matrix $A = \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

(a) Decide whether this economy is productive or not.

(b) Compute the total output vector $\mathbf{x}$, if the final demand vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(c) Which sectors run at profit, if the price vector is $[4, 3]$?

Solution. For (a)-(b), we have to compute the Leontief inverse $(I_2 - A)^{-1}$.

We calculate the inverse of $I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ -0.5 & 0.5 \end{bmatrix}$ using basis transformation (or we could use the cheat sheet for 2×2 matrices, see slide 4/6):

$$\begin{array}{c|cc} & \mathbf{v}_1 & \mathbf{v}_2 \\ \hline \mathbf{e}_1 & 0.2^* & 0 \\ \mathbf{e}_2 & -0.5 & 0.5 \end{array} \to \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{v}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{e}_2 & 2.5 & 0.5^* \end{array} \to \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{v}_2 & 5 & 2 \end{array} \qquad \Longrightarrow \qquad (I_2 - A)^{-1} = \begin{bmatrix} 5 & 0 \\ 5 & 2 \end{bmatrix}.$$

(No rearrangement of rows and columns was needed in the final step.)

Exercise. An economy consists of 2 sectors with input-output matrix $A = \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

(a) Decide whether this economy is productive or not.

(b) Compute the total output vector $\mathbf{x}$, if the final demand vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(c) Which sectors run at profit, if the price vector is $[4, 3]$?

Solution. For (a)-(b), we have to compute the Leontief inverse $(I_2 - A)^{-1}$.

We calculate the inverse of $I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ -0.5 & 0.5 \end{bmatrix}$ using basis transformation (or we could use the cheat sheet for 2×2 matrices, see slide 4/6):

$$\begin{array}{c|cc} & \mathbf{v}_1 & \mathbf{v}_2 \\ \hline \mathbf{e}_1 & 0.2^* & 0 \\ \mathbf{e}_2 & -0.5 & 0.5 \end{array} \rightarrow \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{v}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{e}_2 & 2.5 & 0.5^* \end{array} \rightarrow \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{v}_2 & 5 & 2 \end{array} \implies (I_2 - A)^{-1} = \begin{bmatrix} 5 & 0 \\ 5 & 2 \end{bmatrix}.$$

(a) Since the Leontief inverse exists, and it contains no negative numbers, the economy is productive.

Exercise. An economy consists of 2 sectors with input-output matrix $A = \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

(a) Decide whether this economy is productive or not.

(b) Compute the total output vector $\mathbf{x}$, if the final demand vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(c) Which sectors run at profit, if the price vector is $[4, 3]$?

Solution. For (a)-(b), we have to compute the Leontief inverse $(I_2 - A)^{-1}$.

We calculate the inverse of $I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ -0.5 & 0.5 \end{bmatrix}$ using basis transformation (or we could use the cheat sheet for 2×2 matrices, see slide 4/6):

$$\begin{array}{c|cc} & \mathbf{v}_1 & \mathbf{v}_2 \\ \hline \mathbf{e}_1 & 0.2^* & 0 \\ \mathbf{e}_2 & -0.5 & 0.5 \end{array} \to \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{v}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{e}_2 & 2.5 & 0.5^* \end{array} \to \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{v}_2 & 5 & 2 \end{array} \implies (I_2 - A)^{-1} = \begin{bmatrix} 5 & 0 \\ 5 & 2 \end{bmatrix}.$$

(b) The total output vector is

$$\mathbf{x} = (I_2 - A)^{-1}\mathbf{d} = \begin{bmatrix} 5 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 15 \\ 29 \end{bmatrix}.$$

Exercise. An economy consists of 2 sectors with input-output matrix $A = \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix}$.

(a) Decide whether this economy is productive or not.

(b) Compute the total output vector $\mathbf{x}$, if the final demand vector is $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(c) Which sectors run at profit, if the price vector is $[4, 3]$?

Solution. For (a)-(b), we have to compute the Leontief inverse $(I_2 - A)^{-1}$.

We calculate the inverse of $I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ -0.5 & 0.5 \end{bmatrix}$ using basis transformation (or we could use the cheat sheet for 2×2 matrices, see slide 4/6):

$$\begin{array}{c|cc} & \mathbf{v}_1 & \mathbf{v}_2 \\ \hline \mathbf{e}_1 & 0.2^* & 0 \\ \mathbf{e}_2 & -0.5 & 0.5 \end{array} \quad \to \quad \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{v}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{e}_2 & 2.5 & 0.5^* \end{array} \quad \to \quad \begin{array}{c|cc} & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 5 & 0 \\ \mathbf{v}_2 & 5 & 2 \end{array} \qquad \Longrightarrow \qquad (I_2 - A)^{-1} = \begin{bmatrix} 5 & 0 \\ 5 & 2 \end{bmatrix}.$$

(c) With price vector $\mathbf{p} = [4, 3]$, the profit/loss vector is

$$\mathbf{p} - \mathbf{p}A = [4, 3] - [4, 3] \cdot \begin{bmatrix} 0.8 & 0 \\ 0.5 & 0.5 \end{bmatrix} = [4, 3] - [4.7, 1.5] = [-0.7, 1.5].$$

So the first sector runs at loss (the loss is 0.7 in producing 1 unit of products), the second sector runs at profit (the profit is 1.5 in producing 1 unit of products). $\square$