

Elementary basis transformation

Linear algebra

Lecture 9

Read also: Subsection 5.1 in the lecture notes
(See the Documents folder in Coospace)

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1. Assume that we know the coordinate vector of a vector \mathbf{u} with respect to a basis \mathcal{E} . In this lecture we investigate how the coordinate vector changes when we replace a basis vector in \mathcal{E} to an other vector of the vector space. This is called [elementary basis transformation](#).

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3. In \mathbb{R}^n , the basis \mathcal{E} is usually chosen to be the standard basis, because we know the coordinate vector of \mathbf{u} without calculation with respect to the standard basis: it is the vector \mathbf{u} itself. (Recall from Lecture 6.)

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4. The basis transformation is usually performed on more vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ simultaneously, working in a table. This has many applications: It can be used to evaluate rank of vectors/matrix, find the inverse of a matrix, or solve linear systems, etc.

Theorem 1. Let V be an n -dimensional vector space, and let \mathcal{E} denote a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V . Assume that the vector $\mathbf{w} \in V$ has coordinate vector $[\alpha_1, \dots, \alpha_n]^T$ with respect to \mathcal{E} , i.e. let $\mathbf{w} = \alpha_1 \cdot \mathbf{v}_1 + \dots + \alpha_n \cdot \mathbf{v}_n$.

(a) Then

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \dots, \mathbf{v}_n$$

is a basis in V if and only if $\alpha_i \neq 0$.

(b) Assume that $\alpha_i \neq 0$ holds, and let \mathcal{E}' denote the basis in (a). Let $\mathbf{u} \in V$ be a vector with coordinate vector $[\beta_1, \dots, \beta_n]^T$ with respect to \mathcal{E} . Then the coordinate vector of \mathbf{u} with respect to \mathcal{E}' is

$$\left[\frac{\alpha_i \beta_1 - \beta_i \alpha_1}{\alpha_i}, \dots, \frac{\alpha_i \beta_{i-1} - \beta_i \alpha_{i-1}}{\alpha_i}, \frac{\beta_i}{\alpha_i}, \frac{\alpha_i \beta_{i+1} - \beta_i \alpha_{i+1}}{\alpha_i}, \dots, \frac{\alpha_i \beta_n - \beta_i \alpha_n}{\alpha_i} \right]^T.$$

(c) Assume again that $\alpha_i \neq 0$ holds, and let \mathcal{E}' denote the basis in (a). Then the coordinate vector of \mathbf{v}_i with respect to \mathcal{E}' is

$$\left[-\frac{\alpha_1}{\alpha_i}, -\frac{\alpha_2}{\alpha_i}, \dots, -\frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i}, -\frac{\alpha_{i+1}}{\alpha_i}, -\frac{\alpha_{i+2}}{\alpha_i}, \dots, -\frac{\alpha_n}{\alpha_i} \right]^T.$$

Note that both in (b) and (c) the i 'th coordinate has a different structure than the others.

Proof of (b). We have that

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n, \quad (1)$$

$$\mathbf{u} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n, \quad (2)$$

and we have to express \mathbf{u} as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n$.

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From (1), we can express \mathbf{v}_i in terms of $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n$:

$$\alpha_i \mathbf{v}_i = -\alpha_1 \mathbf{v}_1 - \cdots - \alpha_{i-1} \mathbf{v}_{i-1} + \mathbf{w} - \alpha_{i+1} \mathbf{v}_{i+1} - \cdots - \alpha_n \mathbf{v}_n,$$

and so

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i} \mathbf{v}_1 - \cdots - \frac{\alpha_{i-1}}{\alpha_i} \mathbf{v}_{i-1} + \frac{1}{\alpha_i} \mathbf{w} - \frac{\alpha_{i+1}}{\alpha_i} \mathbf{v}_{i+1} - \cdots - \frac{\alpha_n}{\alpha_i} \mathbf{v}_n \quad (3)$$

(note that $\alpha_i \neq 0$ by the assumptions).

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(note that $\alpha_i \neq 0$ by the assumptions). Then we substitute (3) into (2):

$$\begin{aligned} \mathbf{u} &= \beta_1 \mathbf{v}_1 + \cdots + \beta_{i-1} \mathbf{v}_{i-1} + \beta_i \left(-\frac{\alpha_1}{\alpha_i} \mathbf{v}_1 - \cdots - \frac{\alpha_{i-1}}{\alpha_i} \mathbf{v}_{i-1} + \frac{1}{\alpha_i} \mathbf{w} - \frac{\alpha_{i+1}}{\alpha_i} \mathbf{v}_{i+1} - \cdots - \frac{\alpha_n}{\alpha_i} \mathbf{v}_n \right) \\ &+ \beta_{i+1} \mathbf{v}_{i+1} + \cdots + \beta_n \mathbf{v}_n = \frac{\alpha_i \beta_1 - \beta_i \alpha_1}{\alpha_i} \mathbf{v}_1 + \cdots + \frac{\alpha_i \beta_{i-1} - \beta_i \alpha_{i-1}}{\alpha_i} \mathbf{v}_{i-1} + \\ &\quad \frac{\beta_i}{\alpha_i} \mathbf{v}_i + \frac{\alpha_i \beta_{i+1} - \beta_i \alpha_{i+1}}{\alpha_i} \mathbf{v}_{i+1} + \cdots + \frac{\alpha_i \beta_n - \beta_i \alpha_n}{\alpha_i} \mathbf{v}_n, \end{aligned}$$

and the desired coordinate vector is obtained by reading off the coefficients. \square

Proof of (c).

First proof. In the proof of (b), \mathbf{v}_i is expressed as linear combination of the vectors

$$\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{w}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n,$$

see expression (3):

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i} \mathbf{v}_1 - \dots - \frac{\alpha_{i-1}}{\alpha_i} \mathbf{v}_{i-1} + \frac{1}{\alpha_i} \mathbf{w} - \frac{\alpha_{i+1}}{\alpha_i} \mathbf{v}_{i+1} - \dots - \frac{\alpha_n}{\alpha_i} \mathbf{v}_n \quad (3)$$

and so the desired coordinate vector with respect to \mathcal{E} is indeed

$$\left[-\frac{\alpha_1}{\alpha_i}, -\frac{\alpha_2}{\alpha_i}, \dots, -\frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i}, -\frac{\alpha_{i+1}}{\alpha_i}, -\frac{\alpha_{i+2}}{\alpha_i}, \dots, -\frac{\alpha_n}{\alpha_i} \right]^T.$$

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and so the desired coordinate vector with respect to \mathcal{E} is indeed

$$\left[-\frac{\alpha_1}{\alpha_i}, -\frac{\alpha_2}{\alpha_i}, \dots, -\frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i}, -\frac{\alpha_{i+1}}{\alpha_i}, -\frac{\alpha_{i+2}}{\alpha_i}, \dots, -\frac{\alpha_n}{\alpha_i} \right]^T.$$

□

Second proof. The statement in (c) is a special case of (b) with $\mathbf{u} := \mathbf{v}_i$.

The coordinate vector of \mathbf{v}_i with respect to the basis \mathcal{E} is obviously $[0, 0, \dots, 0, 1, 0, 0, \dots, 0]^T$ where the element 1 is in the i 'th position. [Why?]

So the result in (c) is implied by (b), with $\beta_1 = \dots = \beta_{i-1} = \beta_{i+1} = \dots = \beta_n = 0$ and $\beta_i = 1$.

□

From now on, we use a table (called **basis table**) for storing coordinate vectors in a vector space V , see an example below:

- The **row labels** represent vectors in V that form a **basis** of V .
- The **column labels** represent **arbitrary vectors** in V .
- Each **column** contains the **coordinate vector** of the vector in the column label, with respect to the basis given by the row labels.

	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4
\mathbf{e}_1	1	-1	0	2
\mathbf{e}_2	1	2	3	-1
\mathbf{e}_3	0	5	1	-1

In this example, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis of the vector space, and the second column of the table shows that the coordinate vector of \mathbf{v}_2 with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is $[-1, 2, 5]^T$, i.e.

$$\mathbf{v}_2 = (-1) \cdot \mathbf{e}_1 + 2 \cdot \mathbf{e}_2 + 5 \cdot \mathbf{e}_3.$$

Working with basis tables, an **elementary basis transformation** is the interchange of a row label with a column label (this is permitted only if the new row labels still form a basis!) and the recalculation of the basis table. Theorem 1 describes how this recalculation is done, using the old table (the details are discussed on the next slide).

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Example.

	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4			\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4
\mathbf{e}_1	3	2	-1	5		\mathbf{e}_1	5/3	-2/3	1/3	7/3
\mathbf{e}_2	-5	6	4	7	\rightarrow	\mathbf{e}_2	-9	-2	8	-1
\mathbf{e}_3	2	3*	-2	4		\mathbf{e}_3	2/3	1/3	-2/3	4/3

In this example the vectors \mathbf{e}_3 and \mathbf{v}_2 are interchanged. This is said as “ \mathbf{v}_2 is moved into the basis (in place of \mathbf{e}_3)”.

Note that the interchange of \mathbf{e}_3 and \mathbf{v}_2 is indicated by marking the element 2, lying in the row of \mathbf{e}_3 and column of \mathbf{v}_2 , by a star. This element is called the **pivot** element in the table.

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Now we reformulate Theorem 1 in the language of basis tables ...

Elementary basis transformation. A basis table is given.

- (1) Pick a **non-zero** number a in the table. This will be the pivot element. (To avoid fractions, ± 1 is preferred again.)
- (2) Interchange the row label of the pivot with its column label.
- (3) Write $1/a$ in place of the **pivot element** a .
- (4) Divide by a every other element in the **row of the pivot**. ('Other' = 'not pivot'.)
- (5) Divide by $-a$ every other element in the **column of the pivot**. ('Other' = 'not pivot'.)
- (6) All **elements that are not in the row or column of the pivot** are calculated by the '**rectangle rule**': Let d be an element in the old table, outside the row and column of a . Let b be the element in the intersection of the row of d and the column of a , and let c be the element in the intersection of the column of d and the row of a . And let b and c denote the two other elements. (So the elements a, b, c, d are in the corners of a rectangle, see the figure below). Then in the new basis table write $\frac{ad-bc}{a}$ in place of d .

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 \dots & b & \dots & d & \dots \\
 & \vdots & & \vdots & \\
 \dots & a^* & \dots & c & \dots \\
 & \vdots & & \vdots &
 \end{array}
 \rightarrow
 \begin{array}{ccccc}
 & \vdots & & \vdots & \\
 \dots & & \dots & \frac{ad-bc}{a} & \dots \\
 & \vdots & & \vdots & \\
 \dots & & \dots & & \dots \\
 & \vdots & & \vdots &
 \end{array}$$

A detailed example.

	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4	
\mathbf{e}_1	3	2	-1	5	
\mathbf{e}_2	-5	6	4	7	\rightarrow ?
\mathbf{e}_3	2	3	-2	4	

A detailed example.

	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3	\mathbf{v}_4	
\mathbf{e}_1	3	2	-1	5	
\mathbf{e}_2	-5	6	4	7	\rightarrow ?
\mathbf{e}_3	2	3*	-2	4	

- (1) Pick a **non-zero** number in the table. This will be the pivot element. (To avoid fractions, ± 1 is preferred again. But in this example we choose an other pivot to illustrate the calculations.)

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & & & & \\ \mathbf{e}_2 & & & & \\ \mathbf{v}_2 & & & & \end{array}$$

(2) Interchange the row label of the pivot with its column label.

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & & & & \\ \mathbf{e}_2 & & & & \\ \mathbf{v}_2 & & 1/3 & & \end{array}$$

(3) Take the reciprocal of the pivot element.

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & & & & \\ \mathbf{e}_2 & & & & \\ \mathbf{v}_2 & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \end{array}$$

(4) In the row of pivot, divide every other element by the pivot.

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & & -2/3 & & \\ \mathbf{e}_2 & & -2 & & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

(5) In the column of pivot, divide every other element by the **opposite** of the pivot.

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & & \\ \mathbf{e}_2 & & -2 & & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

(6) To obtain the remaining elements, apply the rectangle rule:

$$\frac{3 \cdot 3 - 2 \cdot 2}{3} = \frac{5}{3}$$

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & 1/3 & \\ \mathbf{e}_2 & & -2 & & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

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$$\frac{3 \cdot 3 - 2 \cdot 2}{3} = \frac{5}{3}, \quad \frac{3 \cdot (-1) - 2 \cdot (-2)}{3} = \frac{1}{3}$$

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & 1/3 & 7/3 \\ \mathbf{e}_2 & & -2 & & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

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A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & 1/3 & 7/3 \\ \mathbf{e}_2 & -9 & -2 & & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

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 \frac{3 \cdot 3 - 2 \cdot 2}{3} &= \frac{5}{3}, & \frac{3 \cdot (-1) - 2 \cdot (-2)}{3} &= \frac{1}{3}, & \frac{3 \cdot 5 - 2 \cdot 4}{3} &= \frac{7}{3}, \\
 \frac{3 \cdot (-5) - 6 \cdot 2}{3} &= -9
 \end{aligned}$$

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & 1/3 & 7/3 \\ \mathbf{e}_2 & -9 & -2 & 8 & \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

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 \frac{3 \cdot (-5) - 6 \cdot 2}{3} &= -9, & \frac{3 \cdot 4 - 6 \cdot (-2)}{3} &= 8
 \end{aligned}$$

A detailed example.

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 \frac{3 \cdot (-5) - 6 \cdot 2}{3} &= -9, & \frac{3 \cdot 4 - 6 \cdot (-2)}{3} &= 8, & \frac{3 \cdot 7 - 6 \cdot 4}{3} &= -1.
 \end{aligned}$$

A detailed example.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 3 & 2 & -1 & 5 \\ \mathbf{e}_2 & -5 & 6 & 4 & 7 \\ \mathbf{e}_3 & 2 & 3^* & -2 & 4 \end{array} \rightarrow \begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{e}_3 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 5/3 & -2/3 & 1/3 & 7/3 \\ \mathbf{e}_2 & -9 & -2 & 8 & -1 \\ \mathbf{v}_2 & 2/3 & 1/3 & -2/3 & 4/3 \end{array}$$

More examples.

(a)

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 1^* & 2 & -1 \\ \mathbf{e}_2 & 2 & -1 & 2 \\ \mathbf{e}_3 & 1 & 1 & 1 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & -1 \\ \mathbf{e}_2 & -2 & -5 & 4 \\ \mathbf{e}_3 & -1 & -1 & 2 \end{array}$$

More examples.

(a)

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 1^* & 2 & -1 \\ \mathbf{e}_2 & 2 & -1 & 2 \\ \mathbf{e}_3 & 1 & 1 & 1 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & -1 \\ \mathbf{e}_2 & -2 & -5 & 4 \\ \mathbf{e}_3 & -1 & -1 & 2 \end{array}$$

(b)

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 2 & 1 & 2 \\ \mathbf{e}_2 & -2^* & 1 & 2 \\ \mathbf{e}_3 & 3 & 0 & 1 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_2 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 1 & 2 & 4 \\ \mathbf{v}_1 & -1/2 & -1/2 & -1 \\ \mathbf{e}_3 & 3/2 & 3/2 & 4 \end{array}$$

Recall from Lecture 3 that the inverse of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I_n$.

Moreover, if $|A| \neq 0$, then there exists a unique inverse matrix A^{-1} . (Otherwise, if $|A| = 0$, there is no inverse of A .)

Recall from Lecture 3 that the inverse of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I_n$.

Moreover, if $|A| \neq 0$, then there exists a unique inverse matrix A^{-1} . (Otherwise, if $|A| = 0$, there is no inverse of A .)

Theorem 2. The inverse of an $n \times n$ matrix A can be calculated by following procedure:

- (1) Write the matrix A into a basis table, label the rows with $\mathbf{e}_1, \dots, \mathbf{e}_n$, and label the columns with $\mathbf{v}_1, \dots, \mathbf{v}_n$ in this order.
- (2) Perform elementary basis transformation n times on this table: Move all the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ into the basis, one by one, replacing the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. **This means that an element can be chosen as pivot only if its row label is \mathbf{e}_i and its column label is \mathbf{v}_j for some indices i, j .** (If not all vectors can be moved into the basis, then A is not invertible.)
- (3) When step (2) is done, reorder the rows of the table so that the row labels are sorted in the natural order $\mathbf{v}_1, \dots, \mathbf{v}_n$, then reorder the columns so that the labels are sorted in the natural order $\mathbf{e}_1, \dots, \mathbf{e}_n$.
- (4) After deleting the labels of rows and columns, the obtained $n \times n$ matrix is the inverse matrix A^{-1} .

Example 1. Find the inverse of $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 5 & 2 & 7 \end{bmatrix}$.

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Solution.

1. Perform EBTs: Move the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the basis, one by one. Keep in mind that the pivot element always must have row label \mathbf{e}_i and column label \mathbf{v}_j for some indices i, j .

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 3 & 1^* & 2 \\ \mathbf{e}_2 & 6 & 2 & 5 \\ \mathbf{e}_3 & 5 & 2 & 7 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{e}_1 & \mathbf{v}_3 \\ \hline \mathbf{v}_2 & 3 & 1 & 2 \\ \mathbf{e}_2 & 0 & -2 & 1^* \\ \mathbf{e}_3 & -1 & -2 & 3 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_2 & 3 & 5 & -2 \\ \mathbf{v}_3 & 0 & -2 & 1 \\ \mathbf{e}_3 & -1^* & 4 & -3 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_2 & 3 & 17 & -11 \\ \mathbf{v}_3 & 0 & -2 & 1 \\ \mathbf{v}_1 & -1 & -4 & 3 \end{array}$$

Example 1. Find the inverse of $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 5 & 2 & 7 \end{bmatrix}$.

Solution.

1. Perform EBTs: Move the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the basis, one by one. Keep in mind that the pivot element always must have row label \mathbf{e}_i and column label \mathbf{v}_j for some indices i, j .

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 3 & 1^* & 2 \\ \mathbf{e}_2 & 6 & 2 & 5 \\ \mathbf{e}_3 & 5 & 2 & 7 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{e}_1 & \mathbf{v}_3 \\ \hline \mathbf{v}_2 & 3 & 1 & 2 \\ \mathbf{e}_2 & 0 & -2 & 1^* \\ \mathbf{e}_3 & -1 & -2 & 3 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_2 & 3 & 5 & -2 \\ \mathbf{v}_3 & 0 & -2 & 1 \\ \mathbf{e}_3 & -1^* & 4 & -3 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_2 & 3 & 17 & -11 \\ \mathbf{v}_3 & 0 & -2 & 1 \\ \mathbf{v}_1 & -1 & -4 & 3 \end{array}$$

2. Rearrange rows and columns, and read off the inverse matrix.

$$\rightsquigarrow \begin{array}{c|ccc} & \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & -1 & -4 & 3 \\ \mathbf{v}_2 & 3 & 17 & -11 \\ \mathbf{v}_3 & 0 & -2 & 1 \end{array} \rightsquigarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \hline \mathbf{v}_1 & -4 & 3 & -1 \\ \mathbf{v}_2 & 17 & -11 & 3 \\ \mathbf{v}_3 & -2 & 1 & 0 \end{array} \implies A^{-1} = \begin{bmatrix} -4 & 3 & -1 \\ 17 & -11 & 3 \\ -2 & 1 & 0 \end{bmatrix}.$$

□

Proof of correctness. The correctness of an inverse matrix calculation can be easily verified by a matrix multiplication:

To check that the inverse of $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 5 & 2 & 7 \end{bmatrix}$ is indeed $\begin{bmatrix} -4 & 3 & -1 \\ 17 & -11 & 3 \\ -2 & 1 & 0 \end{bmatrix}$, all we have to do is to verify that

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 5 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 & -1 \\ 17 & -11 & 3 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

holds.

Example 2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 7 \\ 0 & -1 & 5 \end{bmatrix}$.

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Solution.

1. Perform EBTs: Move the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the basis, one by one. Keep in mind that the pivot element always must have row label \mathbf{e}_i and column label \mathbf{v}_j for some indices i, j .

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 1^* & 2 & 2 \\ \mathbf{e}_2 & 2 & 4 & 7 \\ \mathbf{e}_3 & 0 & -1 & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & 2 \\ \mathbf{e}_2 & -2 & 0 & 3 \\ \mathbf{e}_3 & 0 & -1^* & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & 12 \\ \mathbf{e}_2 & -2 & 0 & 3^* \\ \mathbf{v}_2 & 0 & -1 & -5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 9 & 2 & -4 \\ \mathbf{v}_3 & -2/3 & 0 & 1/3 \\ \mathbf{v}_2 & -10/3 & -1 & 5/3 \end{array}$$

Example 2. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 7 \\ 0 & -1 & 5 \end{bmatrix}$.

Solution.

1. Perform EBTs: Move the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the basis, one by one. Keep in mind that the pivot element always must have row label \mathbf{e}_i and column label \mathbf{v}_j for some indices i, j .

$$\begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{e}_1 & 1^* & 2 & 2 \\ \mathbf{e}_2 & 2 & 4 & 7 \\ \mathbf{e}_3 & 0 & -1 & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & 2 \\ \mathbf{e}_2 & -2 & 0 & 3 \\ \mathbf{e}_3 & 0 & -1^* & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{v}_3 \\ \hline \mathbf{v}_1 & 1 & 2 & 12 \\ \mathbf{e}_2 & -2 & 0 & 3^* \\ \mathbf{v}_2 & 0 & -1 & -5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 9 & 2 & -4 \\ \mathbf{v}_3 & -2/3 & 0 & 1/3 \\ \mathbf{v}_2 & -10/3 & -1 & 5/3 \end{array}$$

2. Rearrange rows and columns, and read off the inverse matrix.

$$\begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 \\ \hline \mathbf{v}_1 & 9 & 2 & -4 \\ \mathbf{v}_2 & -10/3 & -1 & 5/3 \\ \mathbf{v}_3 & -2/3 & 0 & 1/3 \end{array} \rightsquigarrow \begin{array}{c|ccc} & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \hline \mathbf{v}_1 & 9 & -4 & 2 \\ \mathbf{v}_2 & -10/3 & 5/3 & -1 \\ \mathbf{v}_3 & -2/3 & 1/3 & 0 \end{array} \implies A^{-1} = \begin{bmatrix} 9 & -4 & 2 \\ -10/3 & 5/3 & -1 \\ -2/3 & 1/3 & 0 \end{bmatrix}.$$

□

In \mathbb{R}^n , the rank of vectors can be also calculated using basis transformation:

Theorem 3. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^n . The rank of these vectors can be calculated by the following procedure:

- (1) Write the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ into the **columns** of a basis table, label the columns with the labels $\mathbf{v}_1, \dots, \mathbf{v}_k$, and label the rows with the labels $\mathbf{e}_1, \dots, \mathbf{e}_n$. (Here $\mathbf{e}_1, \dots, \mathbf{e}_n$ represents the standard basis of \mathbb{R}^n .)
- (2) Perform elementary basis transformation (EBT) on this table as many times as possible: Move as many vectors from $\mathbf{v}_1, \dots, \mathbf{v}_k$ into the basis as possible, one by one. An element can be chosen as pivot only if its row label is \mathbf{e}_i and its column label is \mathbf{v}_j for some indices i, j . In order to reduce the amount of computation, the column of pivot can be removed after performing an EBT, because we do not need the coordinate vector of the \mathbf{e}_i 's. Stop when all vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ were moved into the basis, or when no pivot element can be chosen (a non-zero number with row label \mathbf{e}_i and column label \mathbf{v}_j for some i, j).
- (3) When step (2) is done, we can obtain the rank of $\mathbf{v}_1, \dots, \mathbf{v}_k$: The rank is equal to the number of vectors that could be moved into the basis, i.e. the number of performed EBTs gives the rank. (We have a freedom in choosing the pivot elements, but the process in step (2) will always stop after the same number of steps.)

Example. Calculate the rank of the following vectors (in \mathbb{R}^3):

$$[2, -1, 3]^T, [1, 1, 2]^T, [0, -3, -1]^T, [3, 0, 5]^T.$$

Example. Calculate the rank of the following vectors (in \mathbb{R}^3):

$$[2, -1, 3]^T, [1, 1, 2]^T, [0, -3, -1]^T, [3, 0, 5]^T.$$

Solution.

Write the given vectors into the columns of a basis table, and by performing EBTs, move as many of these vectors into the basis as possible. The pivot element must always have row label \mathbf{e}_i and column label \mathbf{v}_j for some indices i, j . The column of the pivot can be removed after performing an EBT step.

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 2 & 1^* & 0 & 3 \\ \mathbf{e}_2 & -1 & 1 & -3 & 0 \\ \mathbf{e}_3 & 3 & 2 & -1 & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{v}_2 & 2 & 0 & 3 \\ \mathbf{e}_2 & -3 & -3 & -3 \\ \mathbf{e}_3 & -1^* & -1 & -1 \end{array} \rightarrow \begin{array}{c|cc} & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{v}_2 & -2 & 1 \\ \mathbf{e}_2 & 0 & 0 \\ \mathbf{v}_1 & 1 & 1 \end{array}$$

The process terminates here, because all elements in the row of \mathbf{e}_2 are 0.

We could move 2 vectors into the basis (\mathbf{v}_1 and \mathbf{v}_2), so the rank of the given vectors is 2. \square

As the rank of a matrix can be obtained as the rank of its column vectors, the previous method can be also used to find the rank of a matrix (after adjusting step (1) accordingly):

Theorem 4. The rank of an $n \times k$ matrix A can be calculated by the following procedure:

- (1) Write the matrix A into a basis table, and label the columns with the labels $\mathbf{v}_1, \dots, \mathbf{v}_k$, and label the rows with the labels $\mathbf{e}_1, \dots, \mathbf{e}_n$. (Here $\mathbf{e}_1, \dots, \mathbf{e}_n$ represents the standard basis of \mathbb{R}^n , and $\mathbf{v}_1, \dots, \mathbf{v}_k$ are the columns of A .)
- (2) Perform elementary basis transformation (EBT) on this table as many times as possible: Move as many vectors from $\mathbf{v}_1, \dots, \mathbf{v}_k$ into the basis as possible, one by one. An element can be chosen as pivot only if its row label is \mathbf{e}_i and its column label is \mathbf{v}_j for some indices i, j . In order to reduce the amount of computation, the column of pivot can be removed after performing an EBT, because we do not need the coordinate vector of the \mathbf{e}_i 's. Stop when all vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ were moved into the basis, or when no pivot element can be chosen (a non-zero number with row label \mathbf{e}_i and column label \mathbf{v}_j for some i, j).
- (3) When step (2) is done, we can obtain the rank of A : The rank is equal to the number of vectors that could be moved into the basis, i.e. the number of performed EBTs gives the rank. (We have a freedom in choosing the pivot elements, but the process in step (2) will always stop after the same number of steps.)

Example. Calculate the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 1 & -3 & 0 \\ 3 & 2 & -1 & 5 \end{bmatrix}$$

Example. Calculate the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 1 & -3 & 0 \\ 3 & 2 & -1 & 5 \end{bmatrix}$$

Solution.

Write the matrix A into a basis table, and then proceed in the same way as in calculating rank of vectors:

$$\begin{array}{c|cccc} & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{e}_1 & 2 & 1^* & 0 & 3 \\ \mathbf{e}_2 & -1 & 1 & -3 & 0 \\ \mathbf{e}_3 & 3 & 2 & -1 & 5 \end{array} \rightarrow \begin{array}{c|ccc} & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{v}_2 & 2 & 0 & 3 \\ \mathbf{e}_2 & -3 & -3 & -3 \\ \mathbf{e}_3 & -1^* & -1 & -1 \end{array} \rightarrow \begin{array}{c|cc} & \mathbf{v}_3 & \mathbf{v}_4 \\ \hline \mathbf{v}_2 & -2 & 1 \\ \mathbf{e}_2 & 0 & 0 \\ \mathbf{v}_1 & 1 & 1 \end{array}$$

The process terminates here, because all elements in the row of \mathbf{e}_2 are 0.

We could move 2 vectors into the basis (\mathbf{v}_1 and \mathbf{v}_2), so the rank of the matrix A is 2. □

As the previous applications suggest, basis transformation can be always used when Gaussian elimination is applicable (we will see later that linear systems can be also solved using basis transformation). Both methods have its own advantages and drawbacks.

- It is easier to understand what is happening when working with Gaussian elimination.
- However, basis transformation is more mechanical for most students. (For example, there is no need for interchanging rows.)
- Basis transformation gives more freedom in choosing pivot element, so it is easier to find a ± 1 element. (With Gaussian elimination, we can only choose pivot from the “next” column.)
- On the other side, we cannot use the “multiply a row by a suitable number” trick to avoid fractions in basis transformation.
- etc.