# Homogeneous linear systems. Eigenvalues and eigenvectors. 

## Linear algebra

Lecture 8

Read also: Subsection 4.4 and Chapter 7 in the lecture notes
(See the Documents folder in CooSpace)

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Definition. A linear system is homogeneous if all the constant terms on the right-hand side are zeros, i.e. if the linear system has the form

$$
\left\{\begin{array}{c}
a_{1,1} x_{1}+a_{1,2} x_{2}+\cdots+a_{1, n} x_{n}=0 \\
a_{2,1} x_{1}+a_{2,2} x_{2}+\cdots+a_{2, n} x_{n}=0 \\
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Observation. The zero vector $(0,0,0, \ldots, 0)$ with $n$ components is always a solution to a homogeneous linear system with $n$ variables.

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Observation. The zero vector $(0,0,0, \ldots, 0)$ with $n$ components is always a solution to a homogeneous linear system with $n$ variables.
We can say more:
Thm. The set of solutions of a homogeneous linear system with $n$ variables is a subspace of $\mathbb{R}^{n}$. (The solutions of the system are written in vector form and considered as vectors in $\mathbb{R}^{n}$.)

Definition. The set of solutions of a homogeneous linear system is called the solution space of the system.
(By the previous theorem, the solution space is a subspace of $\mathbb{R}^{n}$, where $n$ is the number of variables.)

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Theorem. Assume that a homogeneous linear system has infinitely many solutions. Then a basis of the solution space can be constructed by the following procedure:

- Solve the linear system (there will be free variables).
- For each free variable, set the free variable in question to 1 , and set all other free variables to 0 (then the values of the basic variables are determined), and the obtained solution vector will be an element of the basis.
- Do this for all free variables, and the obtained vectors form a basis of the solution space. (So if there are $k$ free variables, then the constructed basis has $k$ vectors.)

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Corollary. The dimension of the solution space of a homogeneous linear system is equal to the number of free variables.

Example. Find a basis of the solution space of the homogeneous linear system

$$
\left\{\begin{aligned}
x_{1}+4 x_{2}+2 x_{3}-x_{4} & =0 \\
x_{1}+2 x_{2}+x_{4} & =0 \\
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## Solution.

$$
\left[\begin{array}{cccc|c}
1 & 4 & 2 & -1 & 0 \\
1 & 2 & 0 & 1 & 0 \\
1 & 3 & 1 & 0 & 0
\end{array}\right] \sim \cdots \sim\left[\begin{array}{cccc|c}
1 & 4 & 2 & -1 & 0 \\
0 & -2 & -2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

1. Solving the linear system using Gaussian elimination yields that the free variables are $x_{3}$ and $x_{4}$, and the solutions (written as vectors in $\mathbb{R}^{4}$ ) are $[2 a-3 b,-a+b, a, b]^{T}$, where $a$ and $b$ are arbitrary.

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2. We can obtain a basis in the solution space by the previous theorem:

$$
\begin{aligned}
& a=1, b=0 \quad \rightsquigarrow[2,-1,1,0]^{T} \\
& a=0, b=1 \rightsquigarrow[-3,1,0,1]^{T} .
\end{aligned}
$$

So a basis of the solution space is $[2,-1,1,0]^{T},[-3,1,0,1]^{T}$.

## Worked-out example 1. CLICK HERE

Worked-out example 2. CLICK HERE

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Remark 1. If the number of free variables is 1 , then the basis contains only one vector, and such a vector can be obtained by setting the free variable to 1 . (For example, if the solutions are $[3 a,-5 a, a]^{T}$ where $a$ is arbitrary, then a basis is $[3,-5,1]^{T}$.)

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Remark 2. If there are no free variables, i.e. if the homogeneous linear system has only one solution (the zero vector), then the basis of the solution space is empty (i.e. it contains no vectors).

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Remark 2. If there are no free variables, i.e. if the homogeneous linear system has only one solution (the zero vector), then the basis of the solution space is empty (i.e. it contains no vectors).

Remark 3. If a homogeneous linear system has infinitely many solutions, then its solution space has infinitely many bases, so typically there are many different (correct) answers to these exercises.

Imagine that we solve a homogeneous linear system with $n$ variables and with matrix $A$, using Gaussian elimination. Then the number of basic variables is equal to the number of non-zero rows of the row-echelon form of $A$. But the latter number is the rank of $A$ (see Lecture 7). This means that the number of basic variables is $\operatorname{rank}(A)$, and so the number of free variables is $n-\operatorname{rank}(A)$. Combining this with the previous corollary, we get:

Theorem. Assume that the matrix of a homogeneous linear system with $n$ variables is $A$. Then the dimension of the solution space of the system is $n-\operatorname{rank}(A)$.

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The following theorem justifies the importance of solution spaces:
Theorem. Every subspace of $\mathbb{R}^{n}$ is the solution space of a suitable homogeneous linear system with $n$ variables.

Definition. Let $A$ be a square matrix of size $n \times n$. A real number $\lambda$ is an eigenvalue of $A$ if there exists a non-zero (column) vector $\mathbf{v} \in \mathbb{R}^{n}$ such that $A \cdot \mathbf{v}=\lambda \cdot \mathbf{v}$.
A non-zero vector $\mathbf{v} \in \mathbb{R}^{n}$ is an eigenvector of $A$ if there is a real number $\lambda$ such that $A \cdot \mathbf{v}=\lambda \cdot \mathbf{v}$. (In this case we say that $\mathbf{v}$ is an eigenvector that corresponds to the eigenvalue $\lambda$.)

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Example. 3 is an eigenvalue of the matrix $A=\left[\begin{array}{ccc}3 & 4 & 4 \\ 2 & 5 & 2 \\ -4 & -4 & -1\end{array}\right]$ because, for $\mathbf{v}=[0,-1,1]^{T}$,
$A \cdot \mathbf{v}=\left[\begin{array}{ccc}3 & 4 & 4 \\ 2 & 5 & 2 \\ -4 & -4 & -1\end{array}\right] \cdot\left[\begin{array}{c}0 \\ -1 \\ 1\end{array}\right]=\left[\begin{array}{c}0 \\ -3 \\ 3\end{array}\right]=3 \cdot\left[\begin{array}{c}0 \\ -1 \\ 1\end{array}\right]=3 \cdot \mathbf{v}$.
This also means that $[0,-1,1]^{T}$ is an eigenvector of $A$ (corresponding to the eigenvalue 3 ).

Definition. For an $n \times n$ square matrix

$$
A=\left[\begin{array}{cccc}
a_{1,1} & a_{1,2} & \cdots & a_{1, n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2, n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n, 1} & a_{n, 2} & \cdots & a_{n, n}
\end{array}\right]
$$

the characteristic polynomial of $A$ is the determinant

$$
\left|A-x I_{n}\right|=\left|\begin{array}{cccc}
a_{1,1}-x & a_{1,2} & \cdots & a_{1, n} \\
a_{2,1} & a_{2,2}-x & \cdots & a_{2, n} \\
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where $x$ is a variable.
Note that $A-x I_{n}$ is obtained from $A$ by subtracting $x$ from each element in the main diagonal.

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where $x$ is a variable.
Theorem. The eigenvalues of a square matrix $A$ are precisely the (real) roots of the characteristic polynomial of $A$.

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Solution. 1. The characteristic polynomial of $A$ is

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\begin{aligned}
\left|A-x I_{2}\right| & =\left|\begin{array}{cc}
1-x & 2 \\
-1 & 4-x
\end{array}\right|=(1-x)(4-x)-(-2) \\
& =\left(4-x-4 x+x^{2}\right)+2=x^{2}-5 x+6 .
\end{aligned}
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2. Find the roots of the characteristic polynomial (using the quadratic formula):

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x^{2}-5 x+6=0 \quad \Longleftrightarrow \quad x_{1}=2, x_{2}=3
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So the eigenvalues of $A$ are 2 and 3 .

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More worked-out examples. CLICK HERE

Let $A$ be an $n \times n$ matrix, and $\lambda$ is an eigenvalue of $A$. The set

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\mathcal{U}_{\lambda}:=\left\{\mathbf{v} \in \mathbb{R}^{n}: A \cdot \mathbf{v}=\lambda \cdot \mathbf{v}\right\}
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is called the eigenspace of $A$, corresponding to the eigenvalue $\lambda$. (Observe that $\mathcal{U}_{\lambda}$ consists of all eigenvectors corresponding to $\lambda$, plus the zero vector.)

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Theorem. Let $A$ be an $n \times n$ matrix, and $\lambda$ is an eigenvalue of $A$. Then the eigenspace $\mathcal{U}_{\lambda}$ is a subpace of $\mathbb{R}^{n}$. More precisely, $\mathcal{U}_{\lambda}$ is the solution space of the homogeneous linear system with matrix $A-\lambda I_{n}$.

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This means that the eigenspace $\mathcal{U}_{\lambda}$ can be determined by solving the above homogeneous linear system. And even a basis of $\mathcal{U}_{\lambda}$ can be obtained easily, using the method discussed in the theory of homogeneous linear systems.

Example. Determine the eigenspace $\mathcal{U}_{2}$ corresponding to the eigenvalue $\lambda=2$ of

$$
A=\left[\begin{array}{ccc}
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Solution. 1. The eigenspace $\mathcal{U}_{2}$ is the solution space of the homogeneus linear system with matrix $A-2 I_{3}$ :

$$
\left[\begin{array}{lll|l}
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0
\end{array}\right] .
$$

The solutions of this system are $[a, b, b]^{T}$ where $a$ and $b$ are arbitrary (free variables: $x_{1}, x_{3}$ ), so $\mathcal{U}_{2}=\left\{[a, b, b]^{T}: a, b \in \mathbb{R}\right\}$.

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2. A basis of $\mathcal{U}_{2}: \quad a=1, b=0 \rightsquigarrow[1,0,0]^{T}$

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So a basis of $\mathcal{U}_{2}$ is $[1,0,0]^{T},[0,1,1]^{T}$.

Theorem. Let $A$ be an $n \times n$ matrix, and let $\lambda_{1}, \ldots, \lambda_{r}$ be distinct eigenvalues of $A$ with associated eigenvectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{r}$. Then the vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{r}$ are linearly independent in $\mathbb{R}^{n}$.

Corollary. Assume that an $n \times n$ matrix $A$ has $n$ distinct eigenvalues, $\lambda_{1}, \ldots, \lambda_{n}$, with associated eigenvectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$. Then the vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ form a basis of $\mathbb{R}^{n}$.

