# Solving linear systems using Gaussian elimination

# Linear algebra

Gábor V. Nagy Bolyai Intitute Szeged, 2020. **Definition.** A linear system (with m equations and n variables) is a system of equations of the following form:

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$
  
$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$
  
$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m,$$

where the  $a_{i,j}$ 's are given real numbers (called coefficients), the  $b_i$ 's are given real numbers (called constants), and  $x_1, \ldots, x_n$  are the variables of the linear system.

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**Definition.** An *n*-tuple  $(s_1, \ldots, s_n)$  of real numbers is a solution of the above linear system, if we substitute  $s_1$  for  $x_1$ ,  $s_2$  for  $x_2$ ,  $\ldots$ , and  $s_n$  for  $x_n$ , then for every equation of the system, the left side will equal the right side.

Definition. The matrix of the linear system

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$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$$

is the  $m \times n$  matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix},$$

i.e. the (i, j)-entry of A is the coefficient of  $x_j$  in the i'th equation.

Definition. The augmented matrix of the linear system

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$$
  

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$$
  

$$\vdots$$
  

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m$$

is the  $m \times (n+1)$  matrix

$$[A|\mathbf{b}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}$$

i.e. the augmented matrix can be obtained by inserting  $\mathbf{b} = \begin{bmatrix} v_1 \\ \vdots \\ b_m \end{bmatrix}$  into the matrix of the linear system as last column.

#### Example of an augmented matrix.

$$\begin{cases} 2x_1 + x_2 + 3x_3 = -4 \\ 2x_2 - x_3 = 1 \\ 6x_1 - 5x_2 + 4x_3 = 2 \end{cases} \longleftrightarrow \begin{bmatrix} 2 & 1 & 3 & | & -4 \\ 0 & 2 & -1 & | & 1 \\ 6 & -5 & 4 & | & 2 \end{bmatrix}$$

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The augmented matrix contains all information about the linear system, i.e. the linear system can be reconstructed from its augmented matrix. So we can work with augmented matrices, instead of writing the variables  $x_1, \ldots, x_n$ . (This makes things more terse.) **Definition.** Elementary row operations for linear systems:

- interchange two equations,
- multiply an equation by a nonzero number,
- add a multiple of an equation to another equation,
- delete an equation of the form 0 = 0.

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- interchange two rows,
- multiply a row by a nonzero number,
- add a multiple of a row to another row,
- delete a row whose elements are all 0's.

Elementary row operations on matrices will be denoted by  $\sim\!\!.$ 

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## **Definition.** Elementary row operations for (augmented) matrices:

- interchange two rows,
- multiply a row by a nonzero number,
- add a multiple of a row to another row,
- delete a row whose elements are all 0's.

**Theorem.** After performing elementary row operations, the obtained linear system has precisely the same solutions as the old one.

- the all-zero rows are at the bottom of the matrix, and
- each non-zero row has more leading zeros than the row preceding (above) it.

All-zero row is a row containing only  $0\,{}^{\prime}\text{s},$  non-zero row is a row with at least one nonzero element.

The leading zeros of a (non-zero) row are the 0's preceding the first non-zero entry in the row.

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Lecture 5

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#### Example.

$\begin{bmatrix} 1^* \\ 2 \\ 1 \\ -3 \end{bmatrix}$	3     8     1     -5	-1 1 -4 9	$\begin{bmatrix} 2\\ 3\\ 7\\ 2 \end{bmatrix}$	$-2 \cdot -2 \cdot $	1 0 0 0	$3 2^* -2 4$	$-1 \\ 3 \\ -3 \\ 6$	$\begin{array}{c}2\\-1\\5\\8\end{array}$	$\left( \begin{array}{c} + \\ + \\ + \end{array} \right) - 2 \cdot$
	2	$\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	-1 3 0 0	$\begin{bmatrix} 2 \\ -1 \\ 4^* \\ 10 \end{bmatrix} = \begin{bmatrix} -\frac{10}{4} \end{bmatrix}$	$\sim \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}$	3 2 0 0	-1 3 0 0	$\begin{array}{c}2\\-1\\4\\0\end{array}$	

Sometimes we have to interchange rows:

$$\begin{bmatrix} 0^{?} & 3 & -1 & 2\\ 2 & 8 & 1 & 3\\ 1 & 1 & -4 & 7\\ -3 & -5 & 9 & 2 \end{bmatrix} \stackrel{\bullet}{\leftarrow} \sim \begin{bmatrix} 1^{*} & 1 & -4 & 7\\ 2 & 8 & 1 & 3\\ 0 & 3 & -1 & 2\\ -3 & -5 & 9 & 2 \end{bmatrix} \stackrel{\bullet}{\leftarrow} \stackrel{-2 \cdot}{\leftarrow} +3 \cdot \sim \dots$$

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Another example:

 $\begin{bmatrix} 1^* & 2 & -1 & 3 \\ 2 & 4 & 5 & 7 \\ 3 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{-2} -3 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 7 & 1 \\ 0 & -4 & 5 & -8 \end{bmatrix} \xrightarrow{-3} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -4 & 5 & -8 \\ 0 & -4 & 5 & -8 \\ 0 & 0 & 7 & 1 \end{bmatrix}.$ 

The following procedure, called Gaussian elimination, transforms an arbitrary matrix into row-echelon form:

- Perform the following steps (1st step, 2nd step, ... ) until we reach to a row-echelon form of the matrix.
- In the *i*'th step, find the first (leftmost) column in the actual matrix that contains a non-zero element in or below the *i*'th row. Pick a non-zero element (1 or -1 is preferred) of this column in or below the *i*'th row. By interchanging rows, move the picked element to the *i*'th row and then designate it as the pivot element of the *i*'th row. Finally, zero all elements below this new pivot in its column, by adding suitable multiples of the *i*'th row to the rows below it.

A tip. By multiplying rows, we can avoid fractions (in integer matrices). For example, instead of

$$\begin{bmatrix} 3^* & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{bmatrix} \xrightarrow{-\frac{2}{3}} \xrightarrow{-\frac{2}{3}} \xrightarrow{-\frac{4}{3}} \sim \begin{bmatrix} 3 & 4 & 1 & 1 \\ 0 & 1/3^* & -2/3 & -2/3 \\ 0 & -7/3 & -7/3 & -10/3 \end{bmatrix}$$

we can do

$$\begin{bmatrix} 3^{*} & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{bmatrix} / \cdot 3 \sim \begin{bmatrix} 3^{*} & 4 & 1 & 1 \\ 6 & 9 & 0 & 0 \\ 12 & 9 & -3 & -6 \end{bmatrix} \downarrow^{-2} \downarrow^{-4} \cdot \downarrow^{-4} = \begin{bmatrix} 3 & 4 & 1 & 1 \\ 0 & 1^{*} & -2 & -2 \\ 0 & -7 & -7 & -10 \end{bmatrix} \downarrow^{+7} \sim \begin{bmatrix} 3 & 4 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -21 & -24 \end{bmatrix}.$$

1. If we transform the augmented matrix of a linear system into row-echelon form using Gaussian elimination, then the obtained linear system has exactly the same solutions as the original one, because elementary row operations leave the solutions unchanged.

**2.** The solutions of a linear system can be easily read off when the augmented matrix is in row-echelon form, by proceeding from the bottom non-zero row upward.

This yields a general method for solving linear systems:

**1.** Transform the augmented matrix of a linear system into row-echelon form using Gaussian elimination.

**2.** Read off the solutions from the row-echelon form, by proceeding from the bottom non-zero row upward.

**Example.** Solve the linear system

$$\begin{cases} x_1 + 2x_2 + 5x_3 = -9\\ x_1 - x_2 + 3x_3 = 2\\ 3x_1 - 6x_2 - x_3 = 25. \end{cases}$$

**Example.** Solve the linear system

$$\begin{cases} x_1 + 2x_2 + 5x_3 = -9\\ x_1 - x_2 + 3x_3 = 2\\ 3x_1 - 6x_2 - x_3 = 25. \end{cases}$$

#### Solution.

$$\begin{bmatrix} 1^* & 2 & 5 & | & -9 \\ 1 & -1 & 3 & | & 2 \\ 3 & -6 & -1 & | & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 & | & -9 \\ 0 & -3^* & -2 & | & 11 \\ 0 & -12 & -16 & | & 52 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 & | & -9 \\ 0 & -3 & -2 & | & 11 \\ 0 & 0 & -8 & | & 8 \end{bmatrix}.$$

From bottom to top:

 $(3rd) \qquad -8x_3 = 8 \quad \rightsquigarrow \quad x_3 = -1$ 

(2nd)  $-3x_2 - 2x_3 = 11 \quad \rightsquigarrow \quad -3x_3 + 2 = 11 \quad \rightsquigarrow \quad x_2 = -3$ 

(1st) 
$$x_1 + 2x_2 + 5x_3 = -9 \implies x_1 - 6 - 5 = -9 \implies x_1 = 2$$
  
So the solution is  $(2, -3, -1)$ .

**Definition.** In an augmented matrix of a linear system, we call a row contradicting if it has the form

 $\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & | c \end{bmatrix},$ 

for some non-zero number c.

Observation. Since a contradicting row represents the equation

0 = c

for some constant  $c \neq 0$ , which cannot be satisfied, we have the following:

**Theorem (Case 1).** If a row-echelon form of the augmented matrix of a linear system has a contradicting row, then the linear system has no solution.

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$$\begin{cases} x_1 + x_2 + 2x_3 = 3\\ 4x_1 + 4x_2 + 5x_3 = 6\\ 7x_1 + 7x_2 + 8x_3 = 10 \end{cases}$$

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Solution.

$$\begin{bmatrix} 1^* & 1 & 2 & | & 3 \\ 4 & 4 & 5 & | & 6 \\ 7 & 7 & 8 & | & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 0 & 0 & -3^* & -6 \\ 0 & 0 & -6 & | & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & | & 3 \\ 0 & 0 & -3 & | & -6 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

As the third row is contradicting (it represents the equation 0 = 1), there is no solution.

**Theorem (Case 2).** If a row-echelon form (REF) of the augmented matrix of a linear system has no contradicting rows, and the number of non-zero rows of the REF is equal to the number of variables (i.e. the number of columns on the left side of the vertical bar), then the linear system has exactly one solution, which can be obtained by proceeding from the bottom non-zero row of the REF upward.

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**Example.** The linear system

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on slide 7 provides an example for this scenario. The row echelon form had no contradicting rows, and the REF had 3 non-zero rows (the number of variables is also 3). And there is one solution.

**Theorem (Case 3).** If a row-echelon form (REF) of the augmented matrix of a linear system has no contradicting rows, and the number of non-zero rows of the REF is less than the number of variables (i.e. the number of columns on the left side of the vertical bar), then the linear system has infinitely many solutions, which can be obtained as follows. The variables that correspond to the columns of pivot elements are called basic variables: while the other variables are called free variables. The free variables can be set arbitrarily, and the basic variables can be expressed in terms of free variables, by proceeding from the bottom non-zero row of the REF upward.

Example for Case 3. Consider the linear system

$$\begin{cases} x_1 + 3x_2 - 4x_3 + x_4 = 1\\ 2x_1 + 6x_2 - 7x_3 + x_4 = 6\\ -3x_1 - 9x_2 + 10x_3 - x_4 = -11. \end{cases}$$

Then we have

$$\begin{bmatrix} 1 & 3 & -4 & 1 & | & 1 \\ 2 & 6 & -7 & 1 & | & 6 \\ -3 & -9 & 10 & -1 & | & -11 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 3 & -4 & 1 & | & 1 \\ 0 & 0 & 1 & -1 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

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The row-echelon form has no contradicting rows. The basic variables are  $x_1$  and  $x_3$ , and so the free variables are  $x_2$  and  $x_4$ .

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The row-echelon form has no contradicting rows. The basic variables are  $x_1$  and  $x_3$ , and so the free variables are  $x_2$  and  $x_4$ . We express the basic variables in terms of free variables:

(2nd) 
$$x_3 - x_4 = 4 \quad \rightsquigarrow \quad x_3 = 4 + x_4.$$

(1st)  $x_1 + 3x_2 - 4x_3 + x_4 = 1 \quad \rightsquigarrow$  $x_1 = 1 - 3x_2 + 4(4 + x_4) - x_4 = 17 - 3x_2 + 3x_4.$ 

$$\begin{bmatrix} 1 & 3 & -4 & 1 & | & 1 \\ 2 & 6 & -7 & 1 & | & 6 \\ -3 & -9 & 10 & -1 & | & -11 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 3 & -4 & 1 & | & 1 \\ 0 & 0 & 1 & -1 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The row-echelon form has no contradicting rows. The basic variables are  $x_1$  and  $x_3$ , and so the free variables are  $x_2$  and  $x_4$ . We express the basic variables in terms of free variables: (2nd)  $x_3 - x_4 = 4 \quad \rightsquigarrow \quad x_3 = 4 + x_4$ .

(1st) 
$$x_1 + 3x_2 - 4x_3 + x_4 = 1 \quad \rightsquigarrow$$

 $x_1 = 1 - 3x_2 + 4(4 + x_4) - x_4 = 17 - 3x_2 + 3x_4.$ 

The free variables can take any values:

 $x_2 = a$ ,  $x_4 = b$  (where a and b are arbitrary numbers). Then the values of the basic variables  $x_1, x_3$  are determined by the above calculation. So there are infinitely many solutions: (17 - 3a + 3b, a, 4 + b, b), where a and b are arbitrary numbers. **Conclusion** #1. A linear system has zero, one or infinitely many solutions.

**Conclusion** #2. A linear system has solution(s), if and only if its row-echelon form has no contradicting rows.