Fast determinant evaluation. Matrix inverse, linear systems.

Linear algebra

Lecture 3

Gábor V. Nagy
Bolyai Intitute
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By this approach, evaluation of a 100×100 determinant would be hopeless (in general), even if we are allowed to use computer.

Fortunately, evaluation of an $n \times n$ determinant can be efficiently reduced to the evaluation of one $(n-1) \times (n-1)$ determinant, which makes determinant calculation possible for large matrices in practice (using computers).

Recall from Lecture 2:

Theorem. Let A be a square matrix.

(1) If we multiply a row (or column) of A by a number, then its determinant is multiplied by the same number.

(2) If two rows (or columns) of a determinant are interchanged, then the value of the determinant is multiplied by -1.

(3) If two rows (or columns) of A are identical, then $|A| = 0$.

(4) The value of $|A|$ is unchanged if a multiple of a row is added to another row, or if a multiple of a column is added to another column.

$$\begin{vmatrix} -1 & 3 & 7 & 9 \\ 2 & 4 & 7 & 0 \\ 5 & 8 & -3 & 1 \\ 3 & 2 & 1 & 1 \end{vmatrix} \begin{matrix} \urcorner \\ +2\cdot \\ \hookleftarrow \\ \end{matrix} = \begin{vmatrix} -1 & 3 & 7 & 9 \\ 2 & 4 & 7 & 0 \\ 3 & 14 & 11 & 19 \\ 3 & 2 & 1 & 1 \end{vmatrix}.$$

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Using rule (4), the evaluation of an $n \times n$ determinant can be reduced to the evaluation of one $(n-1) \times (n-1)$ determinant. And then the evaluation of the $(n-1) \times (n-1)$ determinant can be reduced to the evaluation of one $(n-2) \times (n-2)$ determinant in the same way, and so on.

$$\begin{vmatrix} -1 & 2 & 2 & 2 \\ -1 & 1 & 2 & 3 \\ 3 & 0 & 4 & 5 \\ 2 & -4 & -3 & 2 \end{vmatrix} = ?$$

$$\begin{vmatrix} -5 & 2 & 5 & 2 \\ -1 & 1^* & 2 & 3 \\ 3 & 0 & 4 & 5 \\ 2 & -4 & -3 & 2 \end{vmatrix} \begin{matrix} \leftarrow \\ -2\cdot \\ +4\cdot \\ \leftarrow \end{matrix} = \begin{vmatrix} -3 & 0 & 1 & -4 \\ -1 & 1 & 2 & 3 \\ 3 & 0 & 4 & 5 \\ -2 & 0 & 5 & 14 \end{vmatrix}$$

1. Pick a non-zero element in the matrix (which will be referred as pivot element), and use this element to zero all other elements in its column, by applying rule (4) on the previous slide.

- The pivot element is always indicated by a star in our figures.

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- If the pivot element is a, and we want to zero an element c in its column, then we add $-c/a$ times of the row of a to the row of c, because $c + (-c/a)a = 0$.

$$\begin{vmatrix} \ldots\ a^* \ldots\ldots\ldots \\ \ldots\ c\ \ldots\ldots\ldots \end{vmatrix} \big\rfloor -\frac{c}{a}\cdot \quad = \begin{vmatrix} \ldots\ a^* \ldots\ldots\ldots \\ \ldots\ 0\ \ldots\ldots\ldots \end{vmatrix}$$

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- If the pivot element is a, and we want to zero an element c in its column, then we add $-c/a$ times of the row of a to the row of c, because $c + (-c/a)a = 0$.
- So in a determinant with integer entries, the best choice for pivot is a 1 or -1 entry, because then fractions are avoided.

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$$= 1 \cdot \begin{vmatrix} -3 & 1 & -4 \\ 3 & 4 & 5 \\ -2 & 5 & 14 \end{vmatrix}$$

2. Expand the obtained determinant along the column of pivot.
(Only the term corresponding to the pivot will contribute.)

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$$= 1 \cdot \begin{vmatrix} -3 & 1^* & -4 \\ 3 & 4 & 5 \\ -2 & 5 & 14 \end{vmatrix} \begin{matrix} -4\cdot \\ \leftarrow \quad -5\cdot \\ \leftarrow \end{matrix} = 1 \cdot \begin{vmatrix} -3 & 1 & -4 \\ 15 & 0 & 21 \\ 13 & 0 & 34 \end{vmatrix}$$

$$= 1 \cdot (-1) \cdot 1 \cdot \begin{vmatrix} 15 & 21 \\ 13 & 34 \end{vmatrix} = (-1) \cdot (15 \cdot 34 - 21 \cdot 13) = \boxed{-237}.$$

2. Expand the obtained determinant along the column of pivot.
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3. Repeat the procedure for the determinant of smaller size.

$$\begin{vmatrix} -5 & 2 & 5 & 2 \\ -1 & 1^* & 2 & 3 \\ 3 & 0 & 4 & 5 \\ 2 & -4 & -3 & 2 \end{vmatrix} \begin{matrix} \leftarrow \\ -2\cdot \\ +4\cdot \\ \leftarrow \end{matrix} = \begin{vmatrix} -3 & 0 & 1 & -4 \\ -1 & 1 & 2 & 3 \\ 3 & 0 & 4 & 5 \\ -2 & 0 & 5 & 14 \end{vmatrix}$$

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Remark. Instead of zeroing the elements in the column of pivot, one can also zero the elements in the row of pivot, and then expand the obtained the determinant along this row.

Recall that usually $AB \neq BA$ for matrices A, B. But we have:

Claim. If $AB = I_n$ holds for $n \times n$ matrices A, B (where I_n is the identity matrix of size $n \times n$), then $BA = I_n$ also holds.

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Definition. Given an $n \times n$ matrix A. The $n \times n$ matrix B is called the inverse of A, if $AB = BA = I_n$.

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Theorem + Notation. Let A be an $n \times n$ matrix.

(a) If $|A| \neq 0$, then A has exactly one inverse, and it is denoted by A^{-1}.

(b) If $|A| = 0$, then A has no inverse.

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Proof of (b). Assume that $|A| = 0$. Then $AB = I_n$ cannot occur for any $n \times n$ matrix B, because

$$|AB| = |A| \cdot |B| = 0 \cdot |B| = 0,$$

but $|I_n| = 1$. $\square$

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Definition. A square matrix A is called invertible (or nonsingular), if it has an inverse, i.e. if $|A| \neq 0$.

A square matrix A is called singular, if it has no inverse, i.e. if $|A| = 0$.

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Exercise. Show that

$$\begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix}.$$

Definition. A linear system (with m equations and n variables) is a system of equations of the following form:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m, \end{cases}$$

where the $a_{i,j}$'s are given real numbers (called coefficients), the b_i's are given real numbers (called constants), and $x_1, \ldots, x_n$ are the variables of the linear system.

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where the $a_{i,j}$'s are given real numbers (called coefficients), the b_i's are given real numbers (called constants), and $x_1, \ldots, x_n$ are the variables of the linear system.

Definition. An n-tuple $(s_1, \ldots, s_n)$ of real numbers is a solution of the above linear system, if we substitute s_1 for x_1, s_2 for x_2, $\ldots$, and s_n for x_n, then for every equation of the system, the left side will equal the right side.

Example. The system of equations

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 2 \\ 2x_1 + 2x_2 - 5x_3 = -1 \\ -x_1 + 2x_2 + 4x_3 = 11 \end{cases}$$

is a linear system (with 3 equations and 3 variables).

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And $(-1, 3, 1)$ is a solution of it, because

$$(-1) + 2 \cdot 3 - 3 \cdot 1 = 2$$
$$2 \cdot (-1) + 2 \cdot 3 - 5 \cdot 1 = -1.$$
$$-(-1) + 2 \cdot 3 + 4 \cdot 1 = 11$$

We also say that $x_1 = -1,\ x_2 = 3,\ x_3 = 1$ is a solution of the linear system.

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We also say that $x_1 = -1,\ x_2 = 3,\ x_3 = 1$ is a solution of the linear system.

Remark. Solving a linear system means finding all solutions of it.

Definition. The matrix of the linear system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix},$$

i.e. the (i, j)-entry of A is the coefficient of x_j in the i'th equation.

Example. The matrix of the linear system

$$\begin{cases} 2x_1 + x_2 + 3x_3 = -1 \\ 2x_2 - x_3 = 1 \\ 6x_1 - 5x_2 + 4x_3 = 2 \end{cases}$$

is

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 6 & -5 & 4 \end{bmatrix}.$$

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Definition. We say that a linear system with m equations and n variables is regular, if $m = n$ (in other words, the matrix of the linear system is a square matrix) and the matrix of the linear system has non-zero determinant.

Ex. The above linear system is regular, as $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 6 & -5 & 4 \end{vmatrix} = -36.$

Cramer's rule. If the linear system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

is regular, then it has exactly one solution:

$$x_j = \frac{|A^{(j)}|}{|A|} \qquad \text{(for } j = 1, \dots, n\text{)},$$

where $A = [a_{i,j}]_{m \times n}$ is the matrix of the linear system, and $A^{(j)}$ is the matrix obtained from A by replacing its j'th column to

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Example. The linear system

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is regular, because $m = n = 3$ and

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & -5 \\ -1 & 2 & 4 \end{vmatrix} = -6. \qquad \text{(non-zero)}$$

Thus Cramer's rule can be applied to find the unique solution:

$$|A^{(1)}| = \begin{vmatrix} 2 & 2 & -3 \\ -1 & 2 & -5 \\ 11 & 2 & 4 \end{vmatrix} = 6 \quad \rightsquigarrow \quad x_1 = \frac{|A^{(1)}|}{|A|} = -1$$

$$|A^{(2)}| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -5 \\ -1 & 11 & 4 \end{vmatrix} = -18 \quad \rightsquigarrow \quad x_2 = \frac{|A^{(2)}|}{|A|} = 3$$

$$|A^{(3)}| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 11 \end{vmatrix} = -6 \quad \rightsquigarrow \quad x_3 = \frac{|A^{(3)}|}{|A|} = 1.$$

The solution is $(-1, 3, 1)$.

The solution of non-regular linear systems will be discussed in the next lecture.