Determinants

Linear algebra

Lecture 2

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I would like to ask everyone who has been in China in the past three weeks please not to attend the Linear algebra classes on the first and second week.

This request also applies to those students who suspect they might be infected by the coronavirus (for example, they live in common household with someone who has been in China recently, and so on).

Definition. The determinant of a square matrix is a real number associated with the matrix, that is defined as follows. (The determinant of A is denoted by $|A|$, or $\det A$.)

1. If A is a 1×1 matrix, i.e. $A = [\,a_{1,1}\,]$, then $|A| = a_{1,1}$.

2. If A is a 2×2 matrix, i.e. $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, then

$$|A| = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

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Example.

$$|\,18\,| = 18$$

$$\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 1 \cdot 7 - 3 \cdot 5 = -8.$$

As you can see, we write $\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix}$ instead of $\left| \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \right|$.

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3. If A is a 3×3 matrix, i.e. $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$, then

$$|A| = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix},$$

where the 2×2 determinants are computed as defined in **2**.

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3. In other words,

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}.$$

Example.

$$\begin{vmatrix} 2 & -7 & -5 \\ -3 & 9 & 4 \\ -5 & 1 & 8 \end{vmatrix} = 2\begin{vmatrix} 9 & 4 \\ 1 & 8 \end{vmatrix} - (-7)\begin{vmatrix} -3 & 4 \\ -5 & 8 \end{vmatrix} + (-5)\begin{vmatrix} -3 & 9 \\ 5 & 1 \end{vmatrix}.$$

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Example.

$$\begin{aligned} \begin{vmatrix} 2 & -7 & -5 \\ -3 & 9 & 4 \\ -5 & 1 & 8 \end{vmatrix} &= 2 \underbrace{\begin{vmatrix} 9 & 4 \\ 1 & 8 \end{vmatrix}}_{9\cdot 8-4\cdot 1} - (-7) \underbrace{\begin{vmatrix} -3 & 4 \\ -5 & 8 \end{vmatrix}}_{(-3)\cdot 8-4\cdot(-5)} + (-5) \underbrace{\begin{vmatrix} -3 & 9 \\ -5 & 1 \end{vmatrix}}_{(-3)\cdot 1-9\cdot(-5)} \\ &= 2 \cdot 68 - (-7)(-4) + (-5) \cdot 42 \\ &= 136 - 28 - 210 = \boxed{-102}. \end{aligned}$$

4. In general, if A is an $n \times n$ matrix for $n \geq 2$, i.e. if

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix},$$

then

$$|A| = a_{1,1}D_{1,1} - a_{1,2}D_{1,2} + a_{1,3}D_{1,3} - a_{1,4}D_{1,4} + \cdots + (-1)^{n+1}a_{1,n}D_{1,n},$$

where $D_{1,j}$ denotes the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by removing the first row and j'th column.

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where $D_{1,j}$ denotes the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by removing the first row and j'th column.

Example.

$$\begin{vmatrix} 1 & 3 & 7 & -9 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 2 & -4 \\ 3 & 2 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & 2 & 0 \\ 1 & 2 & -4 \\ 2 & 1 & 1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 3 & 1 & 1 \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & -3 & 0 \\ 0 & 1 & -4 \\ 3 & 2 & 1 \end{vmatrix} - (-9) \cdot \begin{vmatrix} 2 & -3 & 2 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot (-36) - 3 \cdot (-12) + 7 \cdot 54 - (-9)(-30)$$

$$= -36 - (-36) + 378 - 270 = \boxed{108}.$$

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$$|A| = a_{1,1}D_{1,1} - a_{1,2}D_{1,2} + a_{1,3}D_{1,3} - a_{1,4}D_{1,4} + \cdots + (-1)^{n+1}a_{1,n}D_{1,n},$$

where $D_{1,j}$ denotes the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by removing the first row and j'th column.

The determinant is only defined for square matrices.

Theorem. Let A be a square matrix.

(1) If we multiply a row (or column) of A by a number, then its determinant is multiplied by the same number.

(2) If two rows (or columns) of a determinant are interchanged, then the value of the determinant is multiplied by -1.

(3) If two rows (or columns) of A are identical, then $|A| = 0$.

(4) The value of $|A|$ is unchanged if a multiple of a row is added to another row, or if a multiple of a column is added to another column.

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Example. (1)

$$\begin{vmatrix} 1 & 3 & 7 & -9 \\ 2 & -3 & 2 & 0 \\ 10 & 2 & 4 & -6 \\ 3 & 2 & 1 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 3 & 7 & -9 \\ 2 & -3 & 2 & 0 \\ 5 & 1 & 2 & -3 \\ 3 & 2 & 1 & 1 \end{vmatrix}$$

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Example. (2)

$$\begin{vmatrix} -1 & 3 & 7 & 9 \\ 2 & 4 & 7 & 0 \\ 5 & 8 & -3 & 1 \\ 3 & 2 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 8 & -3 & 1 \\ 2 & 4 & 7 & 0 \\ -1 & 3 & 7 & 9 \\ 3 & 2 & 1 & 1 \end{vmatrix}.$$

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Example. (3)

$$\begin{vmatrix} 5 & 8 & -3 & 1 \\ 2 & 4 & 7 & 0 \\ 5 & 8 & -3 & 1 \\ 3 & 0 & 1 & 1 \end{vmatrix} = 0.$$

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Example. (4)

$$\begin{vmatrix} -1 & 3 & 7 & 9 \\ 2 & 4 & 7 & 0 \\ 5 & 8 & -3 & 1 \\ 3 & 2 & 1 & 1 \end{vmatrix} \; {+2\cdot} = \begin{vmatrix} -1 & 3 & 7 & 9 \\ 2 & 4 & 7 & 0 \\ 3 & 14 & 11 & 19 \\ 3 & 2 & 1 & 1 \end{vmatrix}.$$

Theorem (Determinant of matrix product).
For any square matrices A and B of the same size,

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Theorem. The determinant of a (lower or upper) triangular matrix is the product of the entries on its main diagonal.

Example.

$$\begin{vmatrix} 5 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 5 & 8 & -3 & 0 \\ 3 & 7 & 1 & 1 \end{vmatrix} = 5 \cdot 4 \cdot (-3) \cdot 1 = -60.$$

Exercise. Try to prove this.

Theorem (Determinant of transpose).

$$|A^T| = |A|.$$

for any square matrix A.

Example.

$$\begin{vmatrix} 5 & 8 & -3 & 1 \\ 2 & 4 & 7 & 0 \\ 4 & 9 & 1 & 6 \\ 3 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 4 & 3 \\ 8 & 4 & 9 & 0 \\ -3 & 7 & 1 & 1 \\ 1 & 0 & 6 & 1 \end{vmatrix}.$$

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Duality principle. If in a statement that is true for arbitrary determinants we interchange the words ‘row’ and ‘column’, then we get a statement that is also true for arbitrary determinants.

Theorem. A determinant can be expanded according to an arbitrary row i: If A is an $n \times n$ matrix, then

$$|A| = (-1)^{i+1} a_{i,1} D_{i,1} + (-1)^{i+2} a_{i,2} D_{i,2} + \cdots + (-1)^{i+n} a_{i,n} D_{i,n},$$

where $a_{i,j}$ is the (i,j)-entry of A, and $D_{1,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by removing the i'th row and j'th column.

Hint. $(-1)^{i+j}$, the sign belonging to the position (i,j), can be read off from a checkerboard array of plus and minus signs, with a plus sign in the upper left corner:

$$\begin{vmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

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Example. By expanding along the second row,

$$\begin{vmatrix} 1 & 5 & -2 \\ 3 & 4 & 0 \\ -3 & 6 & 1 \end{vmatrix} = -3 \begin{vmatrix} 5 & -2 \\ 6 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 5 \\ -3 & 6 \end{vmatrix}$$

$$= -3 \cdot 17 + 4 \cdot (-5) - 0 = -71.$$

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Theorem. A determinant can be expanded according to an arbitrary column j: If A is an $n \times n$ matrix, then

$$|A| = (-1)^{1+j} a_{1,j} D_{1,j} + (-1)^{2+j} a_{2,j} D_{2,j} + \cdots + (-1)^{n+j} a_{n,j} D_{n,j}.$$

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Example. By expanding along the third column,

$$\begin{vmatrix} 1 & 5 & -2 \\ 3 & 4 & 0 \\ -3 & 6 & 1 \end{vmatrix} = (-2) \begin{vmatrix} 3 & 4 \\ -3 & 6 \end{vmatrix} - 0 \begin{vmatrix} 1 & 5 \\ -3 & 6 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= (-2) \cdot 30 - 0 + 1 \cdot (-11) = -71.$$

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The computation in evaluating a determinant can be minimized by expanding along the row or column that contains the most 0's.