Matrices

Linear algebra

Lecture 1

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Szeged, 2020.

I would like to ask everyone who has been in China in the past three weeks please not to attend the Linear algebra classes on the first and second week.

This request also applies to those students who suspect they might be infected by the coronavirus (for example, they live in common household with someone who has been in China recently, and so on).

Course homepage:

http://www.math.u-szeged.hu/~ngaba/linear/

All materials will be available there.

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Seminar: Midterm and final tests (50 pts + 50 pts):

March 18 and May 13.

Additionally, bonus points can be earned during the semester. Make-up for the mid-term and final tests will only be given for unavoidable and documented absences.

Grades:
0 - 50 pts: fail (1)
51 - 62 pts: pass (2)
63 - 75 pts: satisfactory (3)
76 - 87 pts: good (4)
88 - 100 pts: excellent (5).

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Lecture: You need to pass the seminar requirements first. The offered grade for the lecture course is the grade earned on the seminar course. The students can accept this offered grade until the last day of term-time.

The students who reject the offered grade, can take the (written) lecture exam on every week of the exam period. The lecture exam covers the whole course, it contains both theoretical questions and practice exercises. Sample lecture exams will be available on the course's homepage.

Definition. A (real) matrix is a rectangular array of (real) numbers, in square brackets. The numbers in the matrix are called the entries (or elements) of the matrix. (The entry in the i'th row and j'th column of A is called the (i, j)-entry of A.)

$$A = \begin{bmatrix} 2 & 4 & 0 & 1 \\ -5 & 0.2 & 3 & 1 \\ 1/2 & \sqrt{2} & -2 & 0 \end{bmatrix}$$

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Notation. The set of $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$.

Notation. The element in the i'th row and j'th column of matrix A is denoted by $A_{i,j}$.

Example. For the matrix

$$A = \begin{bmatrix} 2 & 4 & 0 & -1 \\ -5 & 0.2 & 8 & 1 \\ 1/2 & \sqrt{2} & -2 & 0 \end{bmatrix}, \qquad A_{2,3} = 8.$$

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Notation. $[a_{i,j}]_{m\times n}$ denotes the $m \times n$ matrix in which the (i,j)-entry is $a_{i,j}$, for all positions (i,j). (Here the $a_{i,j}$'s are given numbers, or $a_{i,j}$ is an expression in terms of i and j.) That is, $[a_{i,j}]_{m\times n}$ denotes the following matrix:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

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Definition. Two matrices A and B are equal (in written $A = B$), if they have the same size and the corresponding entries are equal, i.e. for all positions (i, j),

$$A_{i,j} = B_{i,j}.$$

Example.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

Definition. A square matrix is a matrix with the same number of rows and columns, i.e. it is a matrix of size $n \times n$ for some n.

$$\begin{bmatrix} 3 & 8 & 0 \\ 2 & 3 & 7 \\ 1 & 4 & 1 \end{bmatrix}$$

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Definition. A row matrix (or row vector) is a matrix that has only one row, i.e. it is a matrix of size $1 \times n$ for some n.

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Definition. A column matrix (or column vector) is a matrix that has only one column, i.e. it is a matrix of size $m \times 1$ for some m.

$$\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

Definition. If A and B are matrices of the same size, say $m \times n$, then the sum of A and B (denoted by $A+B$) is also an $m \times n$ matrix, such that

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$

for all positions (i,j). In other words, if

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix},$$

then

$$A+B = \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \cdots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{m,2}+b_{m,2} & \cdots & a_{m,n}+b_{m,n} \end{bmatrix}.$$

$A+B$ is not defined, if A and B have different sizes!

Examples.

$$\begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 1 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -1 \\ 5 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & -1 & 0 & 2 \\ 0 & 3 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -3 & 2 \\ 1 & 3 & 0 & 2 \\ 4 & 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 & 3 \\ 4 & 2 & 0 & 4 \\ 4 & 4 & 11 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & -1 & 0 & 2 \\ 0 & 3 & 6 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 5 \\ 1 & 0 & 2 \\ -4 & 2 & 5 \end{bmatrix} \text{ is not defined.}$$

Definition. If A is an $m \times n$ matrix, and λ is a (real) number, then the scalar multiple λA is the $m \times n$ matrix obtained from A by multiplying each element by λ.

In other words,

$$\text{if } A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \text{ then } \lambda A = \begin{bmatrix} \lambda a_{1,1} & \lambda a_{1,2} & \cdots & \lambda a_{1,n} \\ \lambda a_{2,1} & \lambda a_{2,2} & \cdots & \lambda a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m,1} & \lambda a_{m,2} & \cdots & \lambda a_{m,n} \end{bmatrix}.$$

λA is also written as $\lambda \cdot A$.

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Example.

$$3 \cdot \begin{bmatrix} 1 & 2 & -4 & 0 \\ 2 & 3 & 0 & 2 \\ 1 & 1 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -12 & 0 \\ 6 & 9 & 0 & 6 \\ 3 & 3 & -15 & 12 \end{bmatrix}.$$

Definition. If A is an $m \times n$ matrix, and λ is a (real) number, then the scalar multiple λA is the $m \times n$ matrix obtained from A by multiplying each element by λ.

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λA is also written as $\lambda \cdot A$.

Definition. $-A$ denotes the matrix $(-1)A$, so if

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \text{then } -A = \begin{bmatrix} -a_{1,1} & -a_{1,2} & \cdots & -a_{1,n} \\ -a_{2,1} & -a_{2,2} & \cdots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m,1} & -a_{m,2} & \cdots & -a_{m,n} \end{bmatrix}.$$

Example. Similarly, $A - B$ denotes the matrix $A + (-1)B$, and $2A - 3B$ denotes the matrix $2A + (-3)B$, and so on.

Definition. Let A be an $m \times n$ matrix. The transpose of A, denoted by A^T, is the $n \times m$ matrix in which

$$(A^T)_{i,j} = A_{j,i}$$

for all positions (i, j). In other words, the rows of A^T are just the columns of A in the same order.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 4 \\ -3 & 1 & 6 \\ 5 & 8 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & 8 \\ 4 & 6 & 5 \end{bmatrix}.$$

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Definition. A (square) matrix A is called symmetric if $A^T = A$.

$$\begin{bmatrix} 1 & -2 & 5 & 4 \\ -2 & 3 & 0 & 9 \\ 5 & 0 & 7 & 6 \\ 4 & 9 & 6 & 2 \end{bmatrix}$$

Definition. For matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{r\times s}$, the product AB is defined only if $n = r$, i.e. if the number of columns of A is equal to the number of rows of B.

So assume that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times s}$. Then AB is an $m \times s$ matrix, whose elements are defined as follows:

$$(AB)_{i,j} := A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + A_{i,3}B_{3,j} + \cdots + A_{i,n}B_{n,j},$$

for all positions (i, j).

In other words, the (i, j)-entry of AB is computed in the following way: "Multiply each entry of row i of A by the corresponding entry of column j of B, and add the results."

AB is also written as $A \cdot B$.

Do not forget that usually $AB \neq BA$.

Example.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 7 & 29 \\ -5 & 22 \end{bmatrix}.$$

Example.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 7 & 29 \\ -5 & 22 \end{bmatrix}.$$

Explanation:

$$B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 5 & 8 \\ 7 & \boxed{29} \\ -5 & 22 \end{bmatrix}$$

$$29 = 6 \cdot 4 + (-1) \cdot 3 + 0 \cdot 1 + 4 \cdot 2$$

Example.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 7 & 29 \\ -5 & 22 \end{bmatrix}.$$

Explanation:

$$\begin{array}{cc} & \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} B \\ \underbrace{\begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix}}_{A} & \color{blue}{\begin{bmatrix} 5 & 8 \\ 7 & 29 \\ -5 & 22 \end{bmatrix} AB} \end{array}$$

Example.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 7 & 29 \\ -5 & 22 \end{bmatrix}.$$

Note. However, the product

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & -1 \\ 6 & -1 & 0 & 4 \\ 5 & 1 & -3 & 1 \end{bmatrix}$$

is not defined. So these two matrices serve as an example of the scenario $AB \neq BA$.

Note. Good practice site on matrix operations:

http://matrixcalc.org/en/

Definition. The main diagonal of a square matrix of size $n \times n$ consists of the elements $a_{1,1}, a_{2,2}, a_{3,3}, \ldots, a_{n,n}$.

$$\begin{bmatrix} 1 & -2 & 5 & 4 \\ 3 & 3 & 0 & 9 \\ 6 & 1 & 0 & 6 \\ -1 & 4 & 6 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -2 & 5 & 4 \\ 3 & 3 & 0 & 9 \\ 6 & 1 & 0 & 6 \\ -1 & 4 & 6 & 2 \end{bmatrix}$$

Definition. The zero matrix is a matrix in which all entries are 0. The zero matrix of size $m \times n$ is denoted by $\mathbf{0}_{m\times n}$.

The identity matrix is a square matrix with 1's on the main diagonal and 0's elsewhere. The identity matrix of size $n \times n$ is denoted by I_n.

$$\mathbf{0}_{3\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition. A square matrix is said to be diagonal if each entry outside the main diagonal is 0. A square matrix is said to be lower/upper triangular if all the entries above/below the main diagonal are 0.

Examples.

$$\text{A diagonal matrix: } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

$$\text{A lower triangular matrix: } \begin{bmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & -2 \end{bmatrix}.$$

$$\text{An upper triangular matrix: } \begin{bmatrix} 3 & 2 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Properties of matrix operations. See Theorem 1.12, Theorem 1.14, Theorem 1.17, Theorem 1.18, Theorem 1.21, Theorem 1.29, Theorem 1.30, Theorem 1.32 and Theorem 1.34 in the lecture notes.

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Most of these properties are natural (and easy to prove). Here we present the more difficult or surprising properties only:

1. For all matrices A, B, C such that AB and $(AB)C$ exist,

$$(AB)C = A(BC).$$

2. In most cases (but not always),

$$AB \neq BA.$$

If the order of the factors in a product of matrices changed, then the product may change (or may not exist). For example, it can happen that $ABC \neq ACB$ and $ABC \neq CBA$, etc.

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Most of these properties are natural (and easy to prove). Here we present the more difficult or surprising properties only:

3. For all matrices A, B, such that AB exists,

$$(AB)^T = B^T A^T.$$

On the right-hand side, the order of factors is important: In most cases, $(AB)^T \neq A^T B^T$.