2. Find a basis in the solution space of

$$\begin{cases} x_1 + x_4 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ 2x_2 + x_3 - x_4 = 0 \end{cases}$$

basic variables: x_1, x_2
free variables: x_3, x_4

Sol.: $\left[\begin{array}{cccc|c} 1^* & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 2^* & 1 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$

row-echelon form

(2nd) $2x_2 + x_3 - x_4 = 0 \Rightarrow 2x_2 = -x_3 + x_4 \quad / :2$

$$x_2 = -\frac{1}{2}x_3 + \frac{1}{2}x_4.$$

(1st) $x_1 + x_4 = 0 \Rightarrow x_1 = -x_4$

(free variables) $x_3 = a, \; x_4 = b$

So the solutions are $[-b, -\frac{1}{2}a + \frac{1}{2}b, a, b]^T$, where a, b are arb.

A basis: $a = 1, b = 0 \rightsquigarrow [0, -\frac{1}{2}, 1, 0]^T$

$a = 0, b = 1 \rightsquigarrow [-1, \frac{1}{2}, 0, 1]^T$

Answer: A basis of the solution space is $[0, -\frac{1}{2}, 1, 0]^T, [-1, \frac{1}{2}, 0, 1]^T$.