1. Find a basis in the solution space of the hom. lin. system

$$\begin{cases} x_1+2x_2-x_3+4x_4+x_5=0 \\ x_1+3x_2+2x_3+5x_4+2x_5=0 \\ 2x_1+5x_2+x_3+9x_4+3x_5=0 \end{cases}$$

basic variables $\underbrace{x_1, x_2}$ | free variables: $\underbrace{x_3, x_4, x_5}$

Sol:

$$\left[\begin{array}{ccccc|c} 1^* & 2 & -1 & 4 & 1 & 0 \\ 1 & 3 & 2 & 5 & 2 & 0 \\ 2 & 5 & 1 & 9 & 3 & 0 \end{array}\right] \begin{matrix} \\ - \\ -2\cdot \end{matrix} \sim \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 4 & 1 & 0 \\ 0 & 1^* & 3 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \end{array}\right] \begin{matrix} \\ - \\ \\ \end{matrix} \sim \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 4 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

row-echelon f.

(2nd) $x_2+3x_3+x_4+x_5=0 \Rightarrow \underline{x_2=-3x_3-x_4-x_5}$

(1st) $x_1+2x_2-x_3+4x_4+x_5=0$

$x_1+2(-3x_3-x_4-x_5)-x_3+4x_4+x_5=0$

$x_1-6x_3-2x_4-2x_5-x_3+4x_4+x_5=0 \Rightarrow \underline{x_1=7x_3-2x_4+x_5}$.

The free variables: $\underline{x_3=a}$, $\underline{x_4=b}$, $\underline{x_5=c}$ (where a, b, c are arbitrary).

So the solutions are: $\underline{[7a-2b+c, -3a-b-c, a, b, c]^T}$ (-"-)

A basis: $a=1, b=0, c=0 \rightsquigarrow [7, -3, 1, 0, 0]^T$

$a=0, b=1, c=0 \rightsquigarrow [-2, -1, 0, 1, 0]^T$

$a=0, b=0, c=1 \rightsquigarrow [1, -1, 0, 0, 1]^T$

So a basis of the solution space is

$$\boxed{[7,-3,1,0,0]^T,\ [-2,-1,0,1,0]^T,\ [1,-1,0,0,1]^T.}$$

(So the dimension of the solution space is 3.)