Ex.: Determine the rank of the vectors
$[1,-1,2]^T, [1,1,1]^T, [0,1,2]^T, [1,-2,1]^T$ in $\mathbb{R}^3$.
Decide whether these vectors
a.) are linearly independent;
b.) form a generator system of $\mathbb{R}^3$.
c.) form a basis of $\mathbb{R}^3$.

Solution: Rank calculation:

$$\begin{bmatrix} 1^* & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} \\ - \\ \\ - \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1^* & 2 \\ 0 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -2\cdot \\ + \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5^* \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ +\frac{1}{5}\cdot \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{row-echelon form.}$$

The number of non-zero rows in the row-echelon form is 3. $\Rightarrow$ the rank of vectors is $\underline{\underline{3}}$.
$r = 3$ (rank), $k = 4$ (number of vectors),
$n = 3$ (we are in $\mathbb{R}^4$). So
a.) $r \neq k$ $\Rightarrow$ the vectors are <u>NOT lin. independent</u>.
b.) $r = n$ $\Rightarrow$ the vectors form a <u>generator system</u>.
c.) $r = k = n$ does not hold ($r \neq k \neq n$) $\Rightarrow$
the vectors do <u>NOT</u> form a <u>basis</u>. $\square$