Ex.: Determine the rank of the vectors
$[1,-1,2,1]^T$, $[1,0,1,2]^T$, $[-1,2,-3,1]^T$ in $\mathbb{R}^4$.

Decide whether these vectors

a.) are linearly independent;

b.) form a generator system of $\mathbb{R}^4$;

c.) form a basis of $\mathbb{R}^4$.

Sol.: Rank calculation:

$$\begin{bmatrix} 1^* & -1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 2 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1^* & -1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{R.E.F.}}$$

The number of non-zero rows in the row-echelon form is 3, so the rank of the vectors is 3.

$r = 3$ (rank), $k = 3$ (number of vectors),
$n = 4$ (we are in $\mathbb{R}^4$, the vectors have 4 components).

a.) $r = k \Rightarrow$ the vectors are linearly independent.

b.) $r \neq n \Rightarrow$ the vectors do NOT form a generator system.

c.) $r = k = n$ does not hold ($r = k \neq n$) $\Rightarrow$ the vectors do NOT form a basis.

□