Ex: Find the coordinate vector of $[1,2,1,2]^T$ w.r.t. the basis $[-1,1,1,0]^T, [1,2,1,-1]^T, [1,1,1,1]^T, [-1,-1,-2,-2]^T$ of $\mathbb{R}^4$.

Solution:

$$\left[\begin{array}{cccc|c} -1^* & 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & -2 & 2 \end{array}\right] \begin{matrix} \\ \\ \\ \end{matrix} \begin{matrix} \\ \\ \\ +1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 3 & 2 & -2 & 3 \\ 0 & 2 & 2 & -3 & 2 \\ 0 & -1 & 1 & -2 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 1 \\ 0 & -1^* & 1 & -2 & 2 \\ 0 & 2 & 2 & -3 & 2 \\ 0 & 3 & 2 & -2 & 3 \end{array}\right] \begin{matrix} \\ \\ +2\cdot \\ +3\cdot \end{matrix} \sim \left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -2 & 2 \\ 0 & 0 & 4 & -7 & 6 \\ 0 & 0 & 5 & -8 & 9 \end{array}\right] \sim \quad /\cdot 4$$

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -2 & 2 \\ 0 & 0 & 4^* & -7 & 6 \\ 0 & 0 & 20 & -32 & 36 \end{array}\right] \begin{matrix} \\ \\ \\ -5\cdot \end{matrix} \sim \left[\begin{array}{cccc|c} -1 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -2 & 2 \\ 0 & 0 & 4 & -7 & 6 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right]$$

row-echelon form.

(4th) $3x_4 = 6 \Rightarrow \underline{x_4 = 2}$

(3rd) $4x_3 - 7x_4 = 6 \Rightarrow 4x_3 - 14 = 6 \Rightarrow \underline{x_3 = 5}$

(2nd) $-x_2 + x_3 - 2x_4 = 2 \Rightarrow -x_2 + 5 - 4 = 2 \Rightarrow \underline{x_2 = -1}$

(1st) $-x_1 + x_2 + x_3 - x_4 = 1 \Rightarrow -x_1 - 1 + 5 - 2 = 1$

$\Downarrow$

$\underline{x_1 = 1}$.

So the coordinate vector is $\underline{[1,-1,5,2]^T}$. □