

Ex: Find the coordinate vector of $[1, -1, 1]^T$ with respect to the basis $[1, -1, 3]^T, [2, -1, 4]^T, [3, -1, 4]^T$ of $\mathbb{R}^3$.

Sol: We have to solve the following linear system:

$$\left[\begin{array}{ccc|c} 1^* & 2 & 3 & 1 \\ -1 & -1 & -1 & -1 \\ 3 & 4 & 4 & 1 \end{array} \right] \begin{array}{l} \uparrow + \\ \leftarrow \end{array} -3. \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1^* & 2 & 0 \\ 0 & -2 & -5 & -2 \end{array} \right] \begin{array}{l} \leftarrow +2. \\ \leftarrow \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 \end{array} \right]$$

row-echelon form

$$(3\text{rd}) \quad -x_3 = -2 \Rightarrow \underline{\underline{x_3 = 2}}$$

$$(2\text{nd}) \quad x_2 + 2x_3 = 0$$

$$x_2 + 4 = 0 \Rightarrow \underline{\underline{x_2 = -4}}$$

$$(1\text{st}) \quad x_1 + 2x_2 + 3x_3 = 1$$

$$x_1 - 8 + 6 = 1 \Rightarrow \underline{\underline{x_1 = 3}}$$

So the requested coordinate vector is $\boxed{[3, -4, 2]^T}$.

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