Euler's theorem.

a) A multigraph G has a closed Eulerian trail if and only if G is connected and all degrees in G are even.

b) A multigraph G has a non-closed Eulerian trail if and only if G is connected and precisely two vertices of G have odd degree.

Putting these together:

c) A multigraph G has a Eulerian trail if and only if G is connected and G has 0 or 2 vertices of odd degree.

Complement. The proof of part b) will reveal that if a connected multigraph G has two vertices of odd degree, then one of them must be the first vertex and the other one must be the last vertex of any non-closed Eulerian trail.

Euler's theorem.

a) A multigraph G has a closed Eulerian trail if and only if G is connected and all degrees in G are even.

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Putting these together:

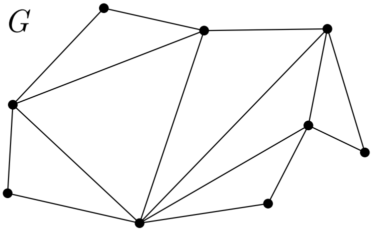
c) A multigraph G has a Eulerian trail if and only if G is connected and G has 0 or 2 vertices of odd degree.

Remark. A multigraph cannot have precisely 1 vertex of odd degree (since the sum of vertex degrees is always even), hence in statement c), the condition "G has 0 or 2 vertices of odd degree" can be rephrased as "G has at most 2 vertices of odd degree".

G has a closed Eulerian trail $\iff G$ is connected and every degree is even.

Proof of a). $\Longrightarrow$ *direction, i.e. necessity of the conditions:*

Assume that the multigraph G has a closed Eulerian trail $\mathcal{T}$.



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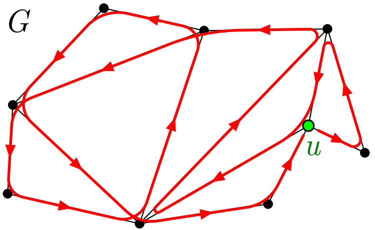

u: start/end vertex

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Assume that the multigraph G has a closed Eulerian trail $\mathcal{T}$.
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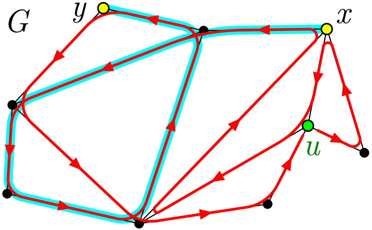

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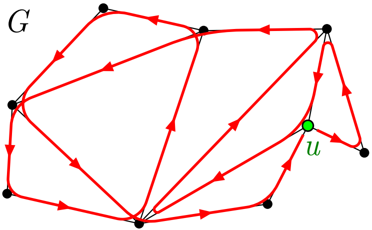
2. When $\mathcal{T}$ visits a vertex, then such a visitation contributes 2 to degree of the vertex (enters/leaves) $\Longrightarrow$ Every vertex degree is even (this is also true for the start/end vertex u). ✓



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In detail:

- If v is vertex different from start/end vertex of $\mathcal{T}$:

Walk through $\mathcal{T}$, and the watch the neighborhood of v ...

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Proof of a). $\implies$ *direction, i.e. necessity of the conditions:*
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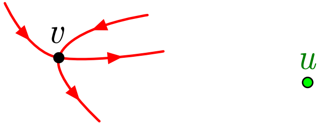
In detail:

- If v is vertex different from start/end vertex of $\mathcal{T}$:

(There is no edge repetition in $\mathcal{T}$.)



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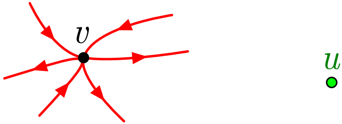

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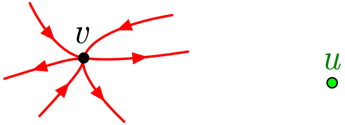
In detail:

- If v is vertex different from start/end vertex of $\mathcal{T}$:

($\mathcal{T}$ traversed every edges, these are all end segments of edges incident to v.)



$d(v) = 2 \times (\text{visitations})$ ← even

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In detail:

- If u is the start/end vertex of $\mathcal{T}$:

– the first edge of $\mathcal{T}$

$$d(u) = 1 +$$

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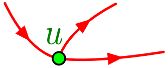
– passing through u



$$d(u) = 1 + 2 \times (\text{number of passes}) +$$

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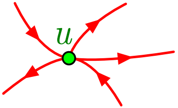

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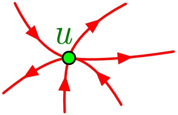
In detail:

- If u is the start/end vertex of $\mathcal{T}$:

– the first edge of $\mathcal{T}$
– passing through u
– the last edge of $\mathcal{T}$



$$d(u) = 1 + 2 \times (\text{number of passes}) + 1 \quad \leftarrow \text{even}$$

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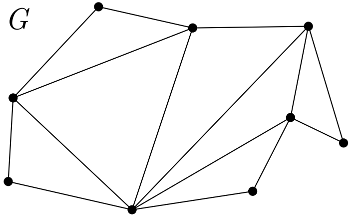

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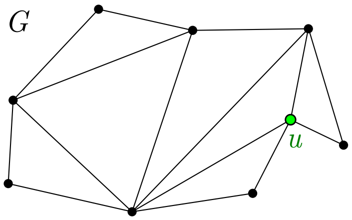
1. Build a trail from an arbitrary start vertex u, in a greedy way.

Keep adding new edges without edge repetition until we stuck ...



If there are more than one free edges to traverse next, pick one arbitrarily.

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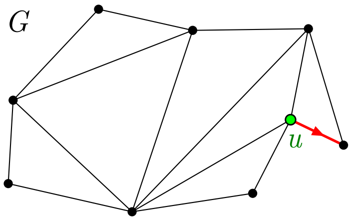

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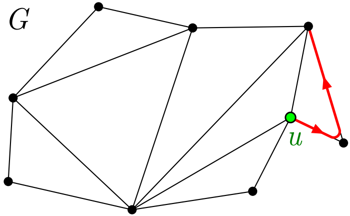

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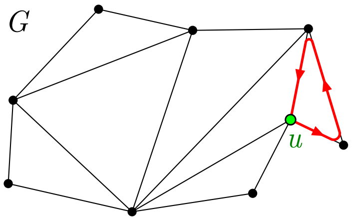

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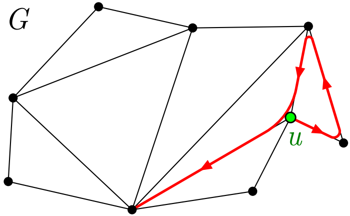

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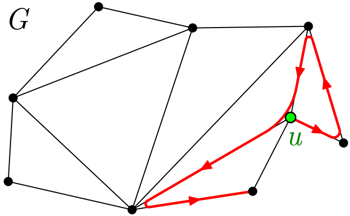

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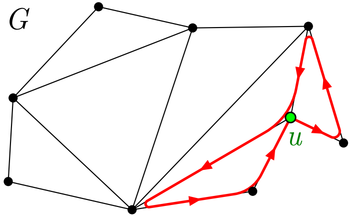

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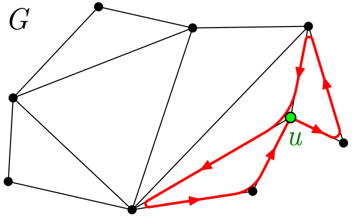
Assume that G is connected and every vertex degree is even.

1. Build a trail from an arbitrary start vertex u, in a greedy way.

Since a trail in G can have at most $|E(G)|$ edges, we will definitely stuck at some point (= all edges starting from the actual vertex have been already traversed).



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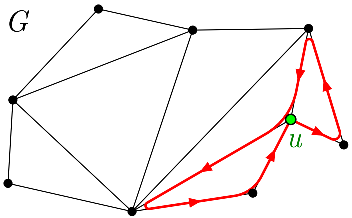
$\forall$ degree is even $\Longrightarrow$ We will stuck at the start vertex u.



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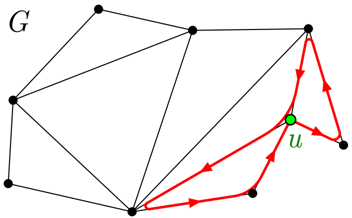
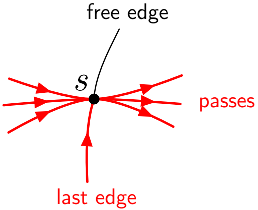
$\forall$ degree is even $\Longrightarrow$ We will stuck at the start vertex u.

REASON: When we stand in a vertex s different from u, then we have passed through this vertex a few times already, and finally entered to that, and so the number of traversed (end segments of) edges around s is odd, so there must be a "free" edge around s which we can use to walk further.





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Assume that G is connected and every vertex degree is even.

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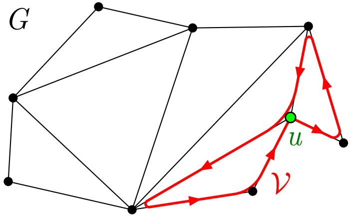
2. When we stuck, we obtain a closed trail $\mathcal{V}$ in G.



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Assume that G is connected and every vertex degree is even.
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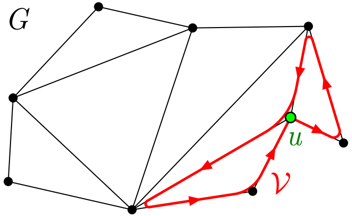
2. When we stuck, we obtain a closed trail $\mathcal{V}$ in G.
3. If this trail $\mathcal{V}$ is Eulerian, then we are done. (Assume it is not.)



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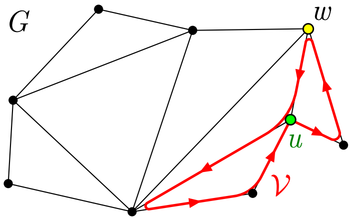
Assume that G is connected and every vertex degree is even.

1. Build a trail from an arbitrary start vertex u, in a greedy way.
2. When we stuck, we obtain a closed trail $\mathcal{V}$ in G.
3. If this trail $\mathcal{V}$ is Eulerian, then we are done. (Assume it is not.)
4. Let w be a vertex on $\mathcal{V}$, which is incident to at least one "black" edge (an edge not in $\mathcal{V}$), too. Since G is connected, such a vertex exists.

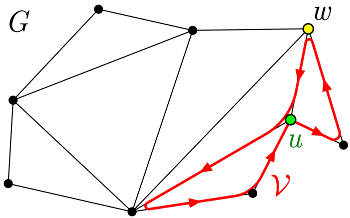


From an arbitrary vertex of $\mathcal{V}$ walk to an arbitrary black edge (we can do this, as G is connected). Consider the first moment when the walk steps onto a black edge.

G has a closed Eulerian trail $\iff$ G is connected and every degree is even.

Proof of a). $\Longleftarrow$ *direction, i.e. sufficiency of the conditions:*

5. Using the above greedy trail building process, build a trail starting from w in the multigraph formed by the BLACK (not yet traversed) edges.



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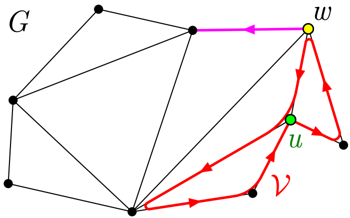

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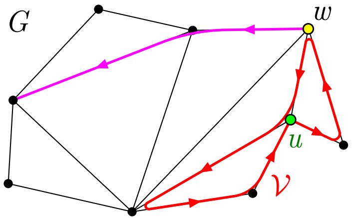

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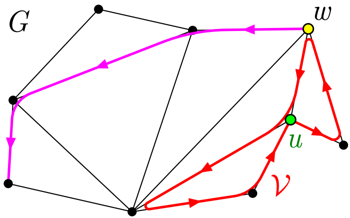

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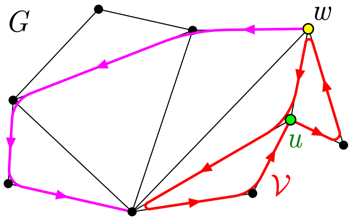

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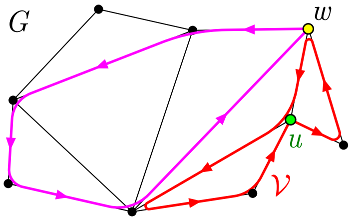
This trail will again terminate at its start vertex w!



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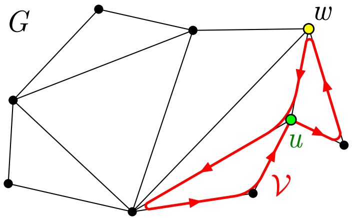

We have already seen that the greedy trail building process always leads to closed trail in a graph with even vertex degrees. (We did not use the connectivity of the graph. This is good news, as the black graph can be disconnected.)

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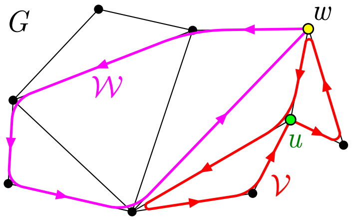

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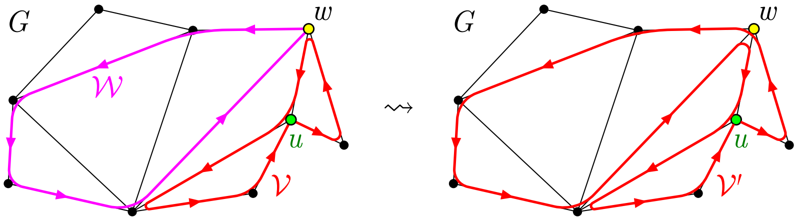
6. The obtained closed trail $\mathcal{W}$ can be inserted to $\mathcal{V}$ at w (see the figure), resulting a longer closed trail $\mathcal{V}'$ in G. (By the choice of w, the trail $\mathcal{W}$ has at least one edge, so $\mathcal{V}'$ is indeed larger.)



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7. Now we reached to the same situation as before: there is a red closed trail in G (now we call it $\mathcal{V}'$, not $\mathcal{V}$). If the closed trail $\mathcal{V}'$ is not yet Eulerian, then we can extend it in the way seen before; the reasoning is the same. And so on, we keep repeating these extension steps until we reach to a closed Eulerian trail (when every edge is traversed by the trail*). □

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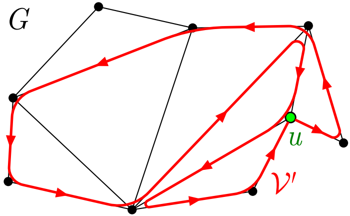

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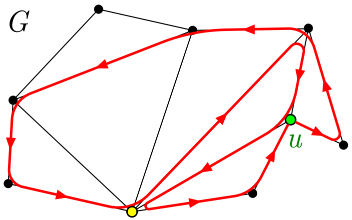

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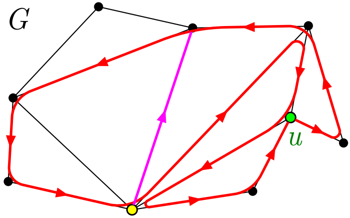

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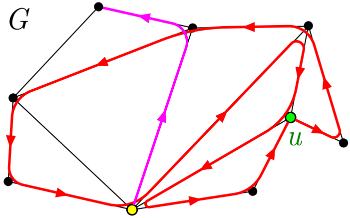

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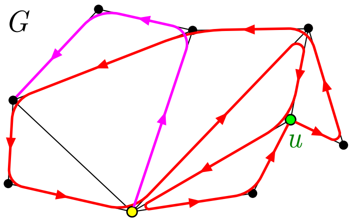

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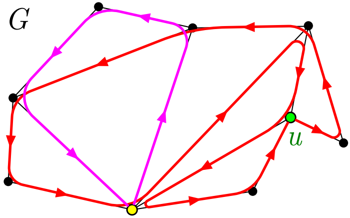

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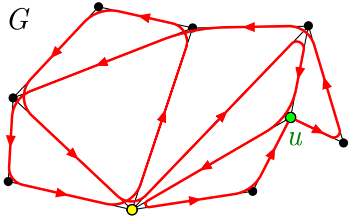

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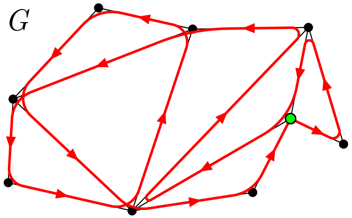

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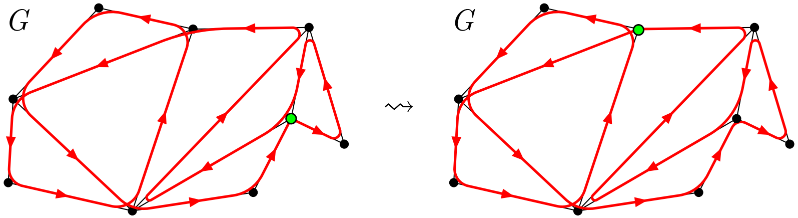

3. A multigraph can have many closed Eulerian trails (in fact, this is the common situation, when the graph satisfies the required properties).

b) G has a non-closed Eulerian trail $\iff$ G is connected and it has precisely two vertices of odd degree.

Proof. The direction „$\Longrightarrow$" is easy, it can be proved analogously to the case a): Here the investigation of degrees gives that the two different end vertices of the Eulerian trail must have odd degree in G, and all other vertices must have even degree.

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The direction „$\Longleftarrow$" can be reduced to the statement a): Let u and v be the two vertices of odd degree in G. Add a new edge e between u and v. (If there is already an edge between u and v in G, then the new edge e will be a parallel edge, which is permitted in multigraphs.) The obtained multigraph G' will be obviously connected, and all of its vertices have even degree, so it contains a closed Eulerian trail $\mathcal{V}$, by statement a). After removing the new edge e from $\mathcal{V}$, the obtained trail will be a non-closed Eulerian trail of the original graph G. $\square$