Euler's theorem.

a) A multigraph G has a closed Eulerian tour if and only if G is connected and all degrees in G are even.

b) A multigraph G has a non-closed Eulerian tour if and only if G is connected and precisely two vertices of G have odd degree.

Putting these together:

c) A multigraph G has a Eulerian tour if and only if G is connected and G has 0 or 2 vertices of odd degree.

Complement. The proof of part b) will reveal that if a connected multigraph G has two vertices of odd degree, then one of them must be the first vertex and the other one must be the last vertex of any non-closed Eulerian tour.

Euler's theorem.

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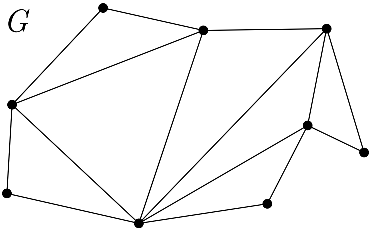
Putting these together:

c) A multigraph G has a Eulerian tour if and only if G is connected and G has 0 or 2 vertices of odd degree.

Remark. A multigraph cannot have precisely 1 vertex of odd degree (since the sum of vertex degrees is always even), hence in statement c), the condition "G has 0 or 2 vertices of odd degree" can be rephrased as "G has at most 2 vertices of odd degree".

G has a closed Eulerian tour $\Longleftrightarrow G$ is connected and every degree is even.

Proof of a). $\Longrightarrow$ *direction, i.e. necessity of the conditions:*
Assume that the multigraph G has a closed Eulerian tour $\mathcal{T}$.



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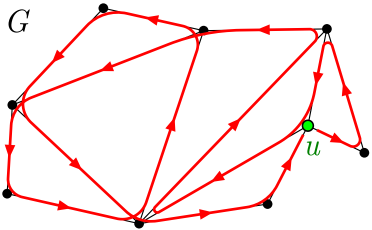

u: start/end vertex

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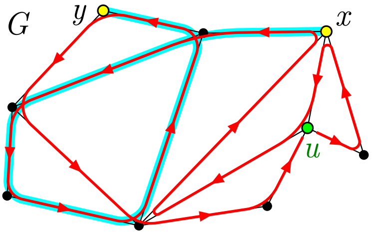

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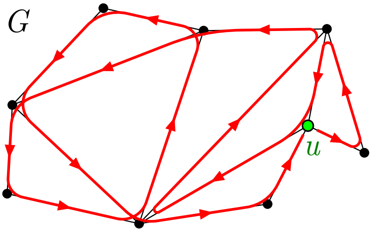

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In detail:

- If v is vertex different from start/end vertex of $\mathcal{T}$:

Walk through $\mathcal{T}$, and the watch the neighborhood of v ...

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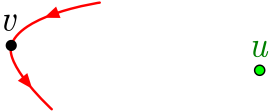

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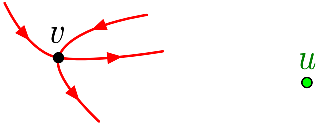
- If v is vertex different from start/end vertex of $\mathcal{T}$:

(There is no edge repetition in $\mathcal{T}$.)



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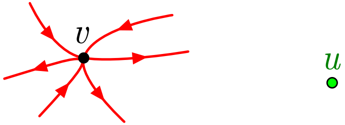

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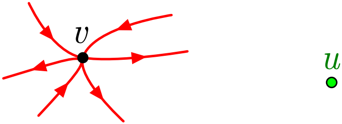
In detail:

- If v is vertex different from start/end vertex of $\mathcal{T}$:

($\mathcal{T}$ traversed every edges, these are all end segments of edges incident to v.)



$d(v) = 2 \times$ (visitations) $\leftarrow$ even

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In detail:

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$$d(u) = 1 +$$

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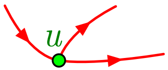
- If u is the start/end vertex of $\mathcal{T}$:

– the first edge of $\mathcal{T}$
– passing through u



$$d(u) = 1 + 2 \times (\text{number of passes}) +$$

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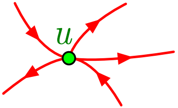

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In detail:

- If u is the start/end vertex of $\mathcal{T}$:

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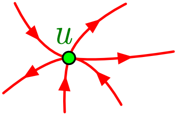
– the last edge of $\mathcal{T}$



$$d(u) = 1 + 2 \times (\text{number of passes}) + 1 \quad \leftarrow \text{even}$$

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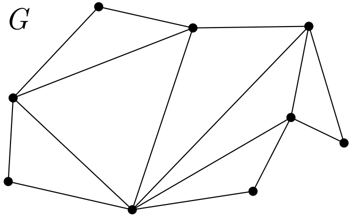

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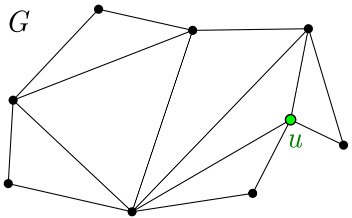
1. Build a tour from an arbitrary start vertex u, in a greedy way.

Keep adding new edges without edge repetition until we stuck ...



If there are more than one free edges to traverse next, pick one arbitrarily.

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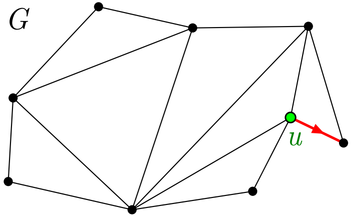

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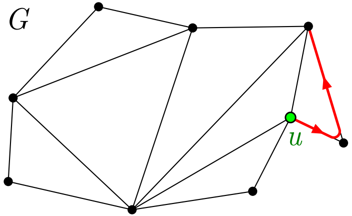

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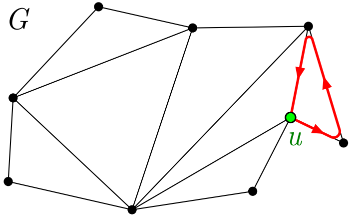

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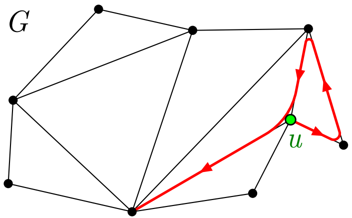

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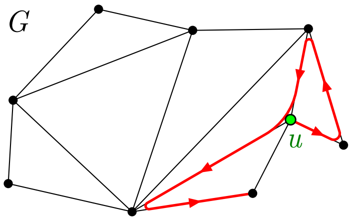

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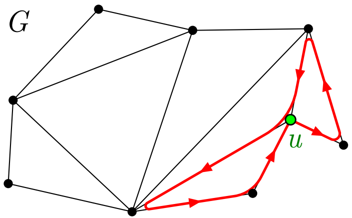

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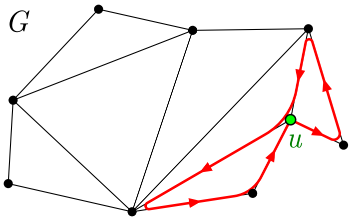
Since a tour in G can have at most $|E(G)|$ edges, we will definitely stuck at some point (= all edges starting from the actual vertex have been already traversed).



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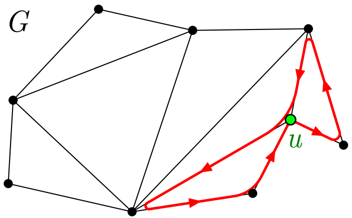
$\forall$ degree is even $\Longrightarrow$ We will stuck at the start vertex u.



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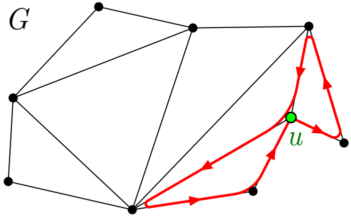
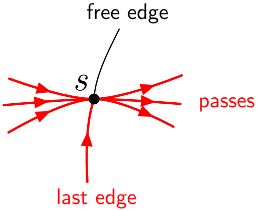
$\forall$ degree is even $\Longrightarrow$ We will stuck at the start vertex u.

REASON: When we stand in a vertex s different from u, then we have passed through this vertex a few times already, and finally entered to that, and so the number of traversed (end segments of) edges around s is odd, so there must be a "free" edge around s which we can use to walk further.





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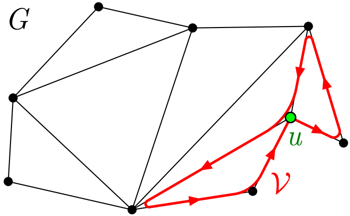
Assume that G is connected and every vertex degree is even.

1. Build a tour from an arbitrary start vertex u, in a greedy way.

2. When we stuck, we obtain a closed tour $\mathcal{V}$ in G.



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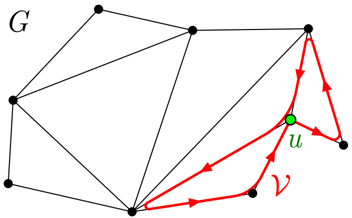
2. When we stuck, we obtain a closed tour $\mathcal{V}$ in G.

3. If this tour $\mathcal{V}$ is Eulerian, then we are done. (Assume it is not.)



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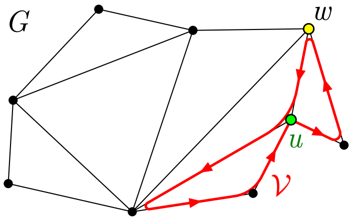
2. When we stuck, we obtain a closed tour $\mathcal{V}$ in G.

3. If this tour $\mathcal{V}$ is Eulerian, then we are done. (Assume it is not.)

4. Let w be a vertex on $\mathcal{V}$, which is incident to at least one "black" edge (an edge not in $\mathcal{V}$), too. Since G is connected, such a vertex exists.



From an arbitrary vertex of $\mathcal{V}$ walk to an arbitrary black edge (we can do this, as G is connected). Consider the first moment when the walk steps onto a black edge.

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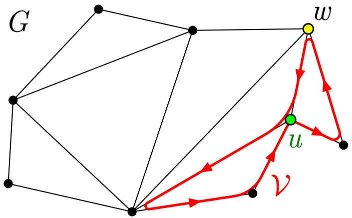
Proof of a). $\Longleftarrow$ *direction, i.e. sufficiency of the conditions:*

5. Using the above greedy tour building process, build a tour starting from w in the multigraph formed by the BLACK (not yet traversed) edges.



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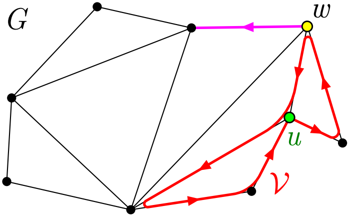

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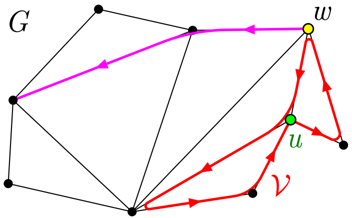

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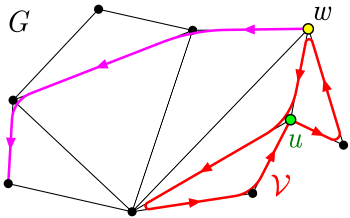

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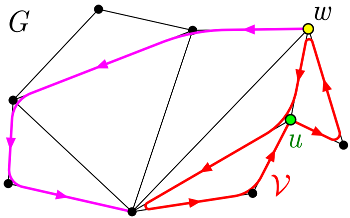

G has a closed Eulerian tour $\Longleftrightarrow$ G is connected and every degree is even.

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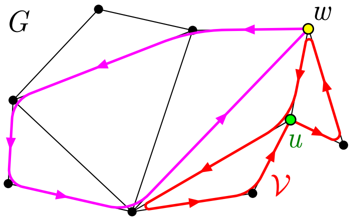
This tour will again terminate at its start vertex w!



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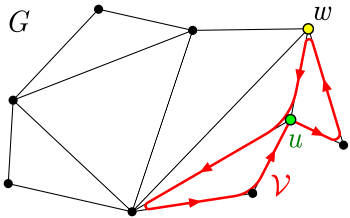

We have already seen that the greedy tour building process always leads to closed tour in a graph with even vertex degrees. (We did not use the connectivity of the graph. This is good news, as the black graph can be disconnected.)

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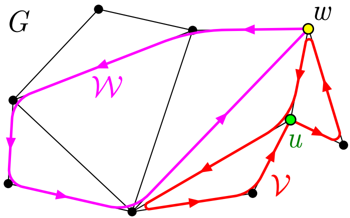

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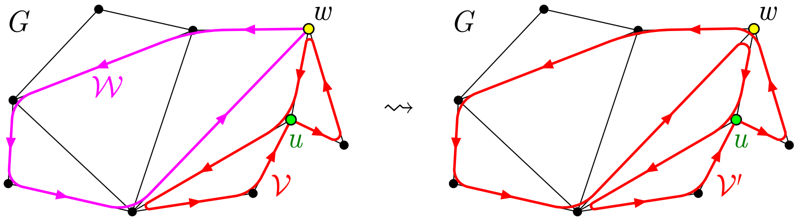
6. The obtained closed tour $\mathcal{W}$ can be inserted to $\mathcal{V}$ at w (see the figure), resulting a longer closed tour $\mathcal{V}'$ in G. (By the choice of w, the tour $\mathcal{W}$ has at least one edge, so $\mathcal{V}'$ is indeed larger.)



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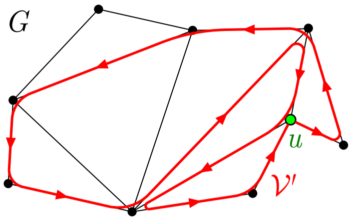
Proof of a). $\Longleftarrow$ *direction, i.e. sufficiency of the conditions:*

7. Now we reached to the same situation as before: there is a red closed tour in G (now we call it $\mathcal{V}'$, not $\mathcal{V}$). If the closed tour $\mathcal{V}'$ is not yet Eulerian, then we can extend it in the way seen before; the reasoning is the same. And so on, we keep repeating these extension steps until we reach to a closed Eulerian tour (when every edge is traversed by the tour*). □

*: and of course the connectivity of G implies that it also visits every vertex.



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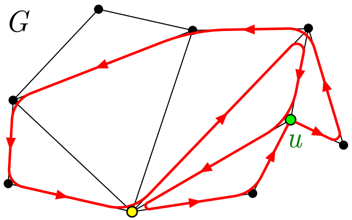

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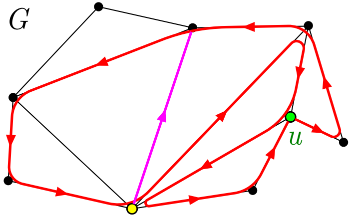

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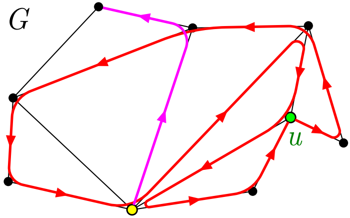

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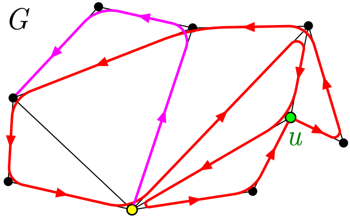

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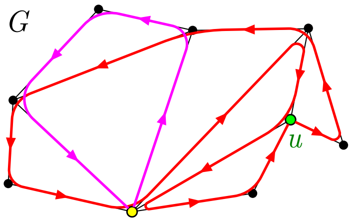

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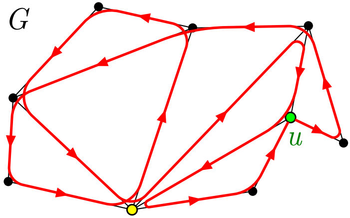

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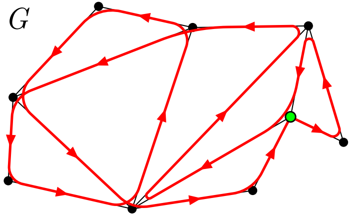

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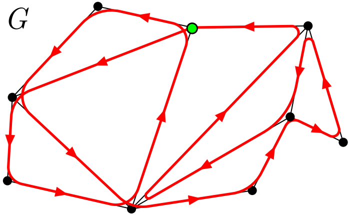

Remarks. 1. We underline that our proof provides an algorithm to find a closed Eulerian tour (if our multigraph has the required properties).

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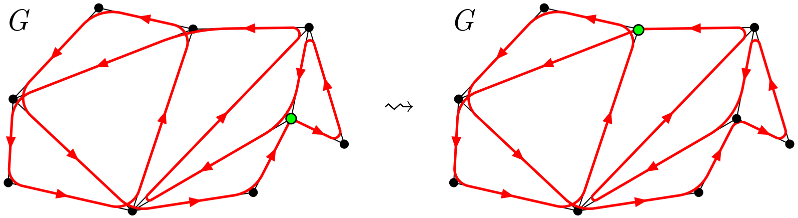

3. A multigraph can have many closed Eulerian tours (in fact, this is the common situation, when the graph satisfies the required properties).

b) G has a non-closed Eulerian tour $\Longleftrightarrow$ G is connected and it has precisely two vertices of odd degree.

Proof. The direction „$\Longrightarrow$" is easy, it can be proved analogously to the case a): Here the investigation of degrees gives that the two different end vertices of the Eulerian tour must have odd degree in G, and all other vertices must have even degree.

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Proof. The direction „$\Longrightarrow$" is easy, it can be proved analogously to the case a): Here the investigation of degrees gives that the two different end vertices of the Eulerian tour must have odd degree in G, and all other vertices must have even degree.

The direction „$\Longleftarrow$" can be reduced to the statement a): Let u and v be the two vertices of odd degree in G. Add a new edge e between u and v. (If there is already an edge between u and v in G, then the new edge e will be a parallel edge, which is permitted in multigraphs.) The obtained multigraph G' will be obviously connected, and all of its vertices have even degree, so it contains a closed Eulerian tour $\mathcal{V}$, by statement a). After removing the new edge e from $\mathcal{V}$, the obtained tour will be a non-closed Eulerian tour of the original graph G. $\square$