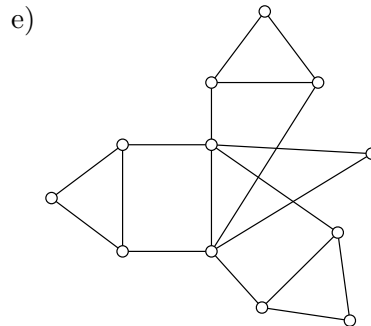
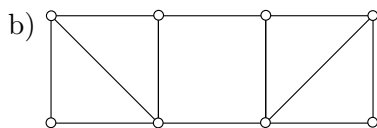
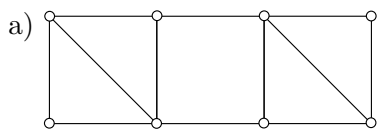


## 7. MATCHINGS IN GENERAL GRAPHS

**Tutte set:** In a graph  $G$ , we call a set  $X \subset V(G)$  *Tutte set* if  $o(G - X) > |X|$ , where  $o(G - X)$  denotes the number of odd components of  $G - X$  (that is, components with an odd number of vertices).

**Tutte theorem:** A graph  $G$  has a perfect matching if and only if there is no Tutte set in  $G$ , i.e. if  $o(G - X) \leq |X|$  for all  $X \subset V(G)$ .

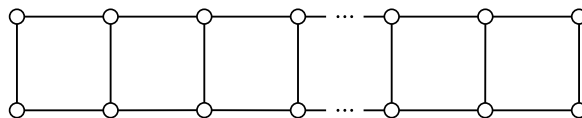
1. Determine the  $\nu$  and  $\tau$  parameters of the following graphs:



- c) Petersen graph,  
d) full binary tree of depth  $k$ .

2. Count the number of perfect matchings in

- a)  $K_n$ ;  
b)  $K_{n,n}$ ;  
c) the graph obtained from  $K_{n,n}$  by removing a perfect matching;  
d) a tree;  
e) the following graph on  $2n$  vertices:



3. Let  $G$  be a simple graph, and let  $M$  be a maximal matching in  $G$ . (A matching  $M$  is maximal, if for each edge  $e$  of the graph,  $M \cup \{e\}$  is not a matching. Such a matching can be easily found by a greedy algorithm.) Prove that

$$\frac{\nu(G)}{2} \leq |M| \leq \nu(G).$$

4. In a  $2n$ -vertex simple graph  $G$ , every vertex degree is at least  $n$ . Prove that there exists a perfect matching in  $G$ . [7.22]

5. Prove that if a matching  $M$  is not of maximum size in a graph  $G$ , then  $M$  can be augmented along a suitable augmenting path.

6. Let  $M_0$  be a matching in a graph  $G$ . Prove that there exists a matching in  $G$  that covers all vertices covered by  $M_0$ . [7.23]

7. Consider a two-player path-building game between Alice and Bob, played on a graph  $G$ . Alice picks a starting vertex  $v$ . They then alternate (starting with Bob) in picking a neighbor of the last vertex. Every vertex may only be chosen once. In this way, they build a path in  $G$  starting at  $v$ . The first player who cannot make a legal move (because there are no more vertices, or because there are no unvisited neighbors) loses. Prove that Bob wins if and only if  $G$  has a perfect matching.

8. Let  $G$  be a simple connected graph with an even number of edges. Using Tutte theorem, prove that the edge set of  $G$  can be partitioned into (edge sets of) paths of length 2.

**9.** Prove Petersen's theorem: Every 2-edge-connected 3-regular graph has a perfect matching. (A graph  $G$  is 2-edge-connected if it is connected and it remains connected after removing one edge arbitrarily.) [7.29]

*Hint:* Use Tutte theorem.

**10.<sup>+</sup>** A graph  $G$  and a function  $f: V(G) \rightarrow \mathbb{N}_0$  are given. Our task is to decide whether there exists a spanning subgraph of  $G$ , in which each vertex  $v$  has degree  $f(v)$ . Reduce this problem to a suitable matching problem (which can be solved by Edmonds' algorithm efficiently).