## 1. Realization of degree sequences

**1.**$^-$ Does there exist a multigraph with degrees
a) $9, 7, 6, 6, 5, 4, 3, 3, 3, 1$
b) $8, 7, 6, 6, 5, 4, 3, 3, 3, 1$?

**2. (Havel–Hakimi algorithm.)** Does there exist a simple graph with degrees
a) $7, 4, 3, 3, 3, 3, 2, 1, 0$
b) $8, 8, 6, 6, 6, 5, 3, 2, 2$
c) $7, 6, 5, 5, 5, 4, 4, 2$
d) $5, 4, 4, 2, 2, 1$?

**3.** Does there exist a *connected* graph with degree sequence $4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1$?

**4.** After a party, every participant tells us how many people (of opposite sex) he or she danced with. We get the following numbers: $9, 9, 9, 9, 6, 6, 6, 5, 3, 3, 3, 3, 3, 3, 3$. Prove that at least one of them is wrong.

**5.** Prove that there exists a $k$-regular graph with $n$ vertices exists if and only if $kn$ is even and $k \le n - 1$.

**6.** In a chess competition involving 10 girls and 20 boys, every girl played exactly 6 games. We know that there were exactly 34 games in which a boy and a girl played against each other. What is the number of "girl vs. girl" games?

**7.** The **Erdős–Gallai theorem** says that the sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ of nonnegative integers can be realized by a simple graph if and only if

$$d_1 + \cdots + d_n \text{ is even}, \quad \text{and} \tag{1}$$

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \quad \text{for all } k \in \{1, \ldots, n\}. \tag{2}$$

Prove that the conditions are necessary.

**8.** Let $n \ge 2$. Prove that the sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers can be realized by a *tree* if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$ and $d_i > 0$ for all $i$.