Qualifying Competition for VJIMC 2016

Category I.

Problem 1. Give all odd, periodic functions $f\colon \mathbb{R} \to \mathbb{R}$ with period 2π which are convex or concave on every closed interval with length π.

Problem 2. Let $f\colon [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Prove that there exist unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.

Problem 3. Prove that the number

$$2^{2^k-1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$.

Problem 4. Suppose that (a_n) is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Show that, the sequence

$$b_n = \frac{\sum_{j=1}^{n} a_j}{n}$$

is convergent and find its limit.

Problem 5. Determine all 2×2 integer matrices A having the following properties:

1. the entries of A are (positive) prime numbers,
2. there exists a 2×2 integer matrix B such that $A = B^2$ and the determinant of B is the square of a prime number.

Problem 6. Let ABC be a non-degenerate triangle in the euclidean plane. Define a sequence $(C_n)_{n=0}^{\infty}$ of points as follows: $C_0 := C$, and C_{n+1} is the center of the incircle of the triangle ABC_n. Find $\lim\limits_{n\to\infty} C_n$.

Problem 7. An unbiased coin is tossed n times. A run is a sequence of throws which result in the same outcome, so that, for example, the sequence HHTHTTH contains five runs. Find the expected number of runs.

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Category II.

Problem 1. Let n closed half-spaces be given in $\mathbb{R}^n$ such that each half-space contains the origin. Prove that their intersection contains a nonzero vector.

Problem 2. Let A and B two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.

Problem 3. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real 10×10 matrices such that $a_{ij} = b_{ij} + 1$ for all i, j and $A^3 = 0$. Prove that $\det(B) = 0$.

Problem 4. Let $[n]$ be the set of first n positive integers. Let $c\colon [n] \to \{r, b\}$ be a red-blue colouring and let $\pi\colon [n] \to [n]$ be a permutation. We say that c is a well-colouring of π if $c(i) = c(\pi(i))$ for every $i \in [n]$. In this case, (c, π) is called a well-coloured permutation on $[n]$.

(a) Show that the number of well-coloured permutations on $[n]$ is $(n+1)!$

(b) Give a bijection between the set of well-coloured permutations of $[n]$ and the set of permutations of $[n+1]$.

Problem 5. Suppose that (a_n) is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Show that, the sequence

$$b_n = \frac{\sum_{j=1}^{n} a_j}{n}$$

is convergent and find its limit.

Problem 6. Let $(G, \cdot)$ be a finite group of order n. Show that every element of G is a square if and only if n is odd.

Problem 7. A biased coin is tossed n times, and heads shows with probability p on each toss. A run is a sequence of throws which result in the same outcome, so that, for example, the sequence HHTHTTH contains five runs. Find the expected number of runs.