8. Extremal graph theory

In the following exercises, all graphs are simple!

1.$^-$ G is a simple graph on n vertices (where $n \geq 4$) such that any two edges of G share a common endpoint. Show that G has at most $n-1$ edges.

2. At most how many edges G can have, if G is simple graph on n vertices that does not contain
 a)$^-$ a cycle;
 b) odd cycle;
 c) triangle;
 d)$^+$ even cycle? [10.1]

3. At most how many edges an n-vertex simple graph G can have, if its chromatic number is k?

4. Let V denote a finite set of n points on a unit circle S^1 in the plane. By a unit circle we understand the set $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.
 a) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.8$?
 b) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.5$?

5. The vectors $\boldsymbol{v_1}, \dots, \boldsymbol{v_n} \in \mathbb{R}^3$ all have length at least 1. Prove that at most $\lfloor n^2/4 \rfloor$ pairs $\{\boldsymbol{v_i}, \boldsymbol{v_j}\}$ have the property $||\boldsymbol{v_i} + \boldsymbol{v_j}|| < 1$.

6. Prove that if the simple graph G has n vertices and no cycles of length shorter than $2k+1$, then the number of edges in G is less than $n(n^{1/k} + 1)$.

7. (Kővári–Sós–Turán theorem.) The simple graph G has no subgraph $K_{r,s}$, where $2 \leq r \leq s$. Prove that G has at most $\frac{\sqrt[r]{s-1}}{2} n^{2-1/r} + \frac{r-1}{2} n = O(n^{2-1/r})$ edges. [10.37]

8. There are given n points on the plane. Prove that the number of (unordered) pairs of points that are unit distance apart is at most $O(n^{3/2})$.

9. Let S be a set of n points in the plane, where $n \geq 3$ and the distance between any two points of S is at least one. Show that no more than $3n - 6$ pairs of points of S can be at distance exactly one.

10. Show that if the chromatic number of G is k, then G contains all trees on k vertices (as subgraphs).

11.$^+$ m identical pizzas are to be shared equally amongst n students. he pizzas are not necessarily divided into equal parts, and the slices of a given pizza may vary in size. Show that this goal can be achieved by dividing the pizzas into a total of $m + n - \gcd(m, n)$ pieces, and no division into a smaller number of pieces will achieve the goal.

12.$^+$ Let T be an arbitrary tree with k edges. The Erdős-T. Sós conjecture states that every simple graph on n vertices not containing T can have at most $\frac{k-1}{2} n$ edges. Prove this conjecture in the special cases when T is a star or path.

13.$^+$ Each of the schools A, B, C has n students. And every student of these schools knows exactly $n+1$ students from the two other schools altogether. Prove that there exist 3 students, one student from each school, such that they mutually know each other.