7. Planar graphs, crossing number

Euler's formula: Let G be a *connected* plane graph on n vertices and e edges, and let f denote the number of faces of G. Then $n - e + f = 2$ holds.

Kuratowski's theorem: A graph G is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Wagner's theorem: A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a graph minor.

1. At an international conference participants from 5 different countries are sitting at a table. Show that there are two people among them whose countries are not neighboring.

2. Prove that if in a convex polyhedron any two vertices are adjacent, then the polyhedron is a tetrahedron.

3. G is a plane graph which contains a Hamiltonian cycle. Prove that the faces of G can be properly colored by 4 colors (without using the four color theorem).

4. If a planar map has even degrees then the faces can be 2-colored in such a way that faces with a common edge on their boundary have different colors. [5.26]

5. Can you draw a planar map in the interior of a pentagon so that the faces are triangles (except, of course, the outermost one), and each point has even degree? [5.28/a]

6. A convex polyhedron has 20 vertices and 12 faces. How many sides do each of the faces have if we know that this number is the same for every face?

7. Prove the following statements:

a) For any simple planar graph G on at least 3 vertices, the inequality $|E(G)| \leq 3|V(G)| - 6$ holds.

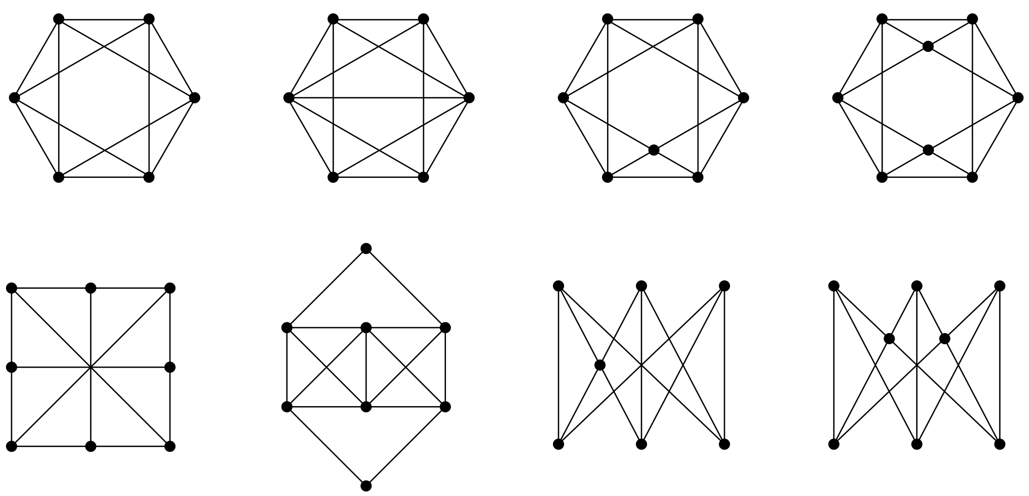
b) For any simple triangle-free planar graph G on at least 3 vertices, the inequality $|E(G)| \leq 2|V(G)| - 4$ holds.

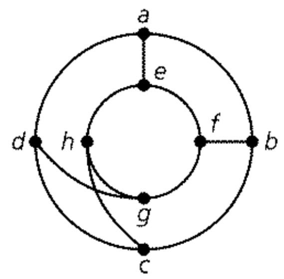
c) If the planar graph G has a cycle, the girth of G is at least g, then we have $|E(G)| \leq \frac{g}{g-2}|V(G)| - \frac{2g}{g-2}$.

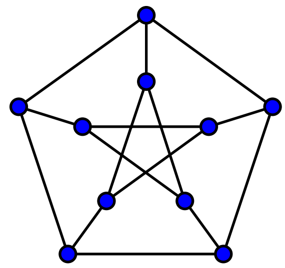
8. Prove that there exist no convex polyhedron whose faces are all hexagons.

9. The vertices of G_8 are the squares of a 8×8 chessboard, and the vertices (squares) u and v are adjacent iff the king can move from u to v (in one move). Decide whether G_8 is planar or not.

10. Which of the following graphs are planar?



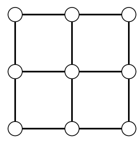




11. a) Determine the crossing number of the graphs in the previous exercise. (It is more difficult for the Petersen graph.)
b) Determine the crossing number of K_6.
c) Determine the crossing number of $K_{4,4}$.

12. Does there exists a simple graph G on 6 vertices, such that neither G nor $\overline{G}$ is planar?

13. Prove that the following graph has a K_4-minor.



14. Prove that if x, y, z are three different vertices of an n-vertex simple planar graph, then

$$d(x) + d(y) + d(z) \leq 2n + 2.$$

15. Which of the following statements are true?
a) If H is topological subgraph of G, then H is a minor of G.
b) If H is minor of G, then H is a topological subgraph of G.