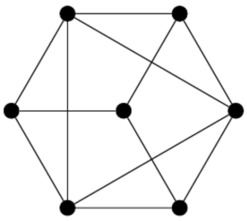
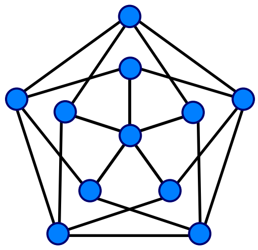
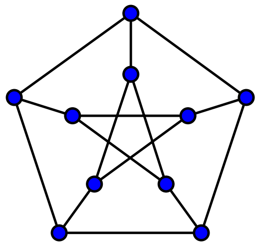
6. Edge coloring

Vizing's theorem: If G is a *simple* graph, then $\chi_e(G) \le \Delta(G) + 1$. (So for simple graphs, $\chi_e(G) = \Delta(G)$ or $\chi_e(G) = \Delta(G) + 1$.)

1. Find the edge chromatic number of the following graphs.

a) 

b) 

c) 

d) the graph obtained from the cycle C_9 by connecting every vertex to its two second neighbors on the cycle,

e) the complete graph K_n.

2. a) Show that if a 3-regular graph have a Hamiltonian cycle, then the edge chromatic number of this graph is 3.
b) Does the Petersen graph have a Hamiltonian cycle?
c) G is a loopless 3-regular graph whose edge-chromatic number is 3, and there is exactly one proper edge coloring of G, apart from permuting the colors. Show that G contains a Hamiltonian cycle.

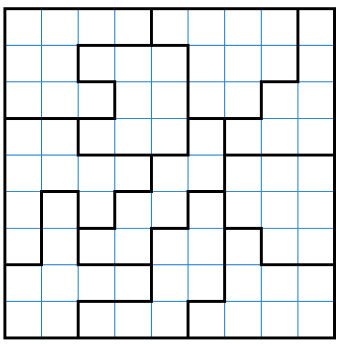
3. G is a 3-regular connected simple graph, which has an edge e such that the graph $G - e$ is disconnected. Prove that $\chi_e(G) = 4$.

4. Let G be a simple graph. The graph $G^{(2)}$ is obtained from two vertex-disjoint copies of G by connecting the vertices that are "clones" of each other (so $|V(G)|$ new edges are added this way). Prove that the edge chromatic number of $G^{(2)}$ is $\Delta(G) + 1$.

5. a) Prove that the edge chromatic number of a d-regular *bipartite* graph is d.
b) Prove that if G is a **bipartite graph**, then $\chi_e(G) = \Delta(G)$.

6. The graph G has a proper edge coloring using k colors. Prove that G also has a proper edge coloring using k colors in which every color appears $\left\lfloor \frac{|E(G)|}{k} \right\rfloor$ or $\left\lceil \frac{|E(G)|}{k} \right\rceil$ times.

7.$^+$ In the Bolyai Sudoku the 9×9 square is divided into regions in a such a way that every region consists of at most 9 cells. The regions can have any shape, one exaple is shown in the figure. Prove that the cells of the Bolyai Sudoku can be filled with the numbers 1-9 such that each row and each column contains pairwise distinct elements (and there is no such requirement for columns).



8.$^+$ Prove that $\chi_e(G) \le 3 \left\lceil \frac{\Delta(G)}{2} \right\rceil$ for any loopless graph G.

Hint. You can apply exercise III/13. (Prove the problem for $2k$-regular graphs first.)
Remark. This problem is close to Shannon's theorem, which states that $\chi_e(G) \le \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor$.

9.$^{+}$ Prove that the edge set of K_{2n+1} can be partitioned into n (edge sets of) Hamiltonian cycles of K_{2n+1}.