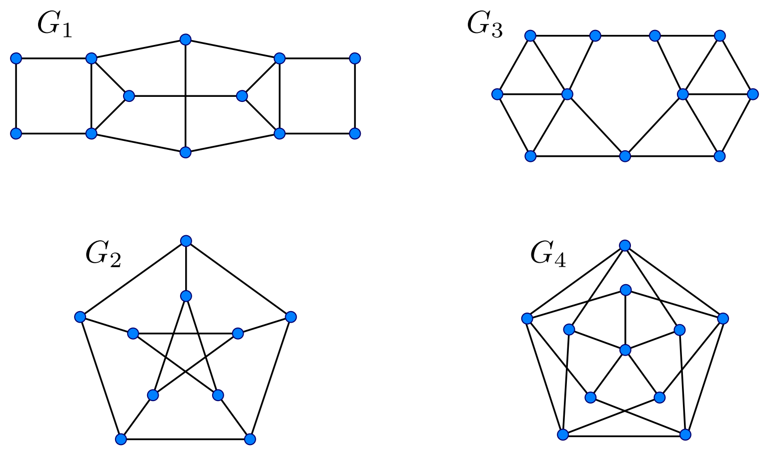
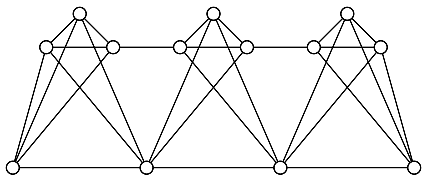
5. Vertex coloring, girth, random method

1. Determine the chromatic number of the following graphs:



2. Determine the chromatic number of the following graph. Use Hajós construction to provide a proof of the lower bound.



3. Without using Hajós' theorem, prove that every non-2-colorable graph is Hajós-constructible from K_3's.

4. The Kneser graph $KG(n,k)$ is defined as follows. The vertex set of $KG(n,k)$ is $\binom{[n]}{k}$, and two vertices (subsets) are adjacent if and only of they are disjoint. Prove that $\chi(KG(n,k)) \leq n-2k+2$, if $n \geq 2k$.
Remark. Using a topological argument, László Lovász proved that equality holds here.

5. (Task scheduling.)

a) In a transport company there are n tasks to perform. Each task t is represented by a closed interval I_t describing the time in which it needs to be performed by some employee. (For instance, task t must be run from 13:00 to 15:00, etc.) Each task is performed by exactly one employee, and the same employee cannot perform two tasks s and t if their intervals I_s and I_t intersect. Construct an efficient algorithm for assinging the given tasks to the employees requiring the least number of employees possible.

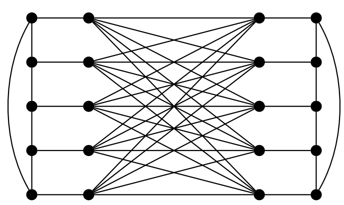
b) How is this problem related to graph coloring?

6. Prove that if G is a bipartite graph, then $\chi(G) = \omega(G)$ and $\chi(\overline{G}) = \omega(\overline{G})$.

7. A graph G is k-critical, if $\chi(G) = k$, but $\chi(H) < k$ holds for all proper subgraph H of G.

a) Determine all 3-critical graphs. [9.17/a]

b) Let n be an odd integer. Connect each partite class of the complete bipartite graph $K_{n,n}$ to a cycle of length n as shown in the figure. Show that the obtained graph is 4-critical. [9.17/b]



c) Prove that in a k-critical graph, every vertex degree is at least $k-1$.

8. The edge set of a graph G can be partitioned into three (disjoint) classes, such that each class forms a bipartite graph. Prove that $\chi(G) \leq 8$.

9. In a simple graph G, any two odd cycles have a common vertex. Prove that $\chi(G) \leq 5$.

10. Determine the girth of the Petersen graph.

11. Show that if a k-regular graph has girth 4, than the graph has at least $2k$ vertices.

12. Prove that if an n-vertex graph has minimum degree $\delta > 2$, then the girth of the graph is at most $2 \log_{\delta-1} n + 2$.

13.$^+$ There are 2000 coins lying on an enormous table. (The coins are not all of the same size.) Some of them might touch each other, but they don't overlap. Show that you can always choose 500 of them such that no two chosen coins touch each other.

14.$^+$ Assume that G has a proper vertex coloring in which every color is assigned to at least two vertices. Show that G has such a coloring with $\chi(G)$ colors, too. [9.4]

15. Let V be a finite set, and let $R_1, \dots, R_m$ be m-element subsets of V. Prove that if $m < 2^{n-1}$, then the elements of V can be colored by 2 colors such that every R_i has elements with distinct colors.

16. Prove that the edges of the complete graph K_n can be colored by 2 colors such that it containts at most

$$\binom{n}{a} 2^{1-\binom{a}{2}}$$

monochromatic clique of size a.