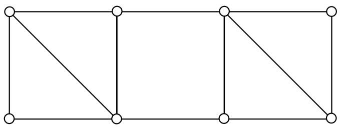
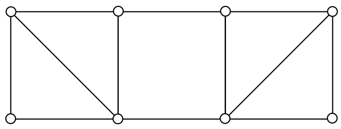
4. Matchings in general graphs

Tutte set: In a graph G, we call a set $X \subset V(G)$ *Tutte set* if $c_1(G - X) > |X|$, where $c_1(G - X)$ denotes the number of odd components of $G - X$ (that is, components with an odd number of vertices).

Tutte theorem: A graph G has a perfect matching if and only if there is no Tutte set in G, i.e. if $c_1(G - X) \leq |X|$ for all $X \subset V(G)$.

1. Determine the ν and τ parameters of the following graphs:

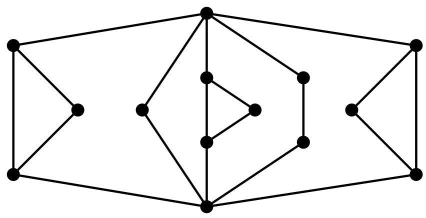
a) 

b) 

c) Petersen graph

d) the $(2^{k+1} - 1)$-vertex complete binary tree of depth k.

2. Determine the ν and τ parameters of the following graph.



3. Count the number of perfect matchings in

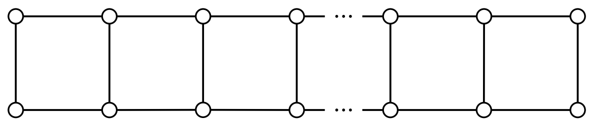
a) K_n;

b) $K_{n,n}$;

c) the graph obtained from $K_{n,n}$ by removing a perfect matching;

d) a tree;

e) the following graph on $2n$ vertices:



4. Let G be a simple graph, and let M be a maximal matching in G. (A matching M is maximal, if for each edge e of the graph, $M \cup \{e\}$ is not a matching. Such a matching can be easily found by a greedy algorithm.) Prove that

$$\frac{\nu(G)}{2} \leq |M| \leq \nu(G).$$

5. In a $2n$-vertex simple graph G, every vertex degree is at least n. Prove that there exists a perfect matching in G. [7.22]

6. Prove that if a matching M is not of maximum size in a graph G, then M can be augmented along a suitable augmenting path.

7. Let M_0 be a matching in a graph G. Prove that there exists a matching of maximum size in G that covers all vertices covered by M_0. [7.23]

8. Consider a two-player path-building game between Alice and Bob, played on a graph G. Alice picks a starting vertex v. They then alternate (starting with Bob) in picking a neighbor of the last vertex. Every vertex may only be chosen once. In this way, they build a path in G starting at v. The first player who cannot make a legal move (because there are no more vertices, or because there are no unvisited neighbors) loses. Prove that Bob wins if and only if G has a perfect matching.

9. Let G be a simple connected graph with an even number of edges. Using Tutte theorem, prove that the edge set of G can be partitioned into (edge sets of) paths of length 2.

10. Prove Petersen's theorem: Every 2-edge-connected 3-regular graph has a perfect matching. (A graph G is 2-edge-connected if it is connected and it remains connected after removing one edge arbitrarily.) [7.29]
Hint: Use Tutte theorem.

11.$^+$ A graph G and a function $f\colon V(G) \to \mathbb{N}_0$ are given. Our task is to decide whether there exists a spanning subgraph of G, in which each vertex v has degree $f(v)$. Reduce this problem to a suitable matching problem (which can be solved by Edmonds' algorithm efficiently).