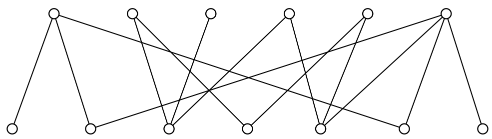
3. Matchings in bipartite graphs

Marriage theorem (Kőnig–Hall and Kőnig–Frobenius theorems). Let G be a bipartite graph with partite classes A and B.

a) G contains a matching covering A if and only if $|N(X)| \geq |X|$ holds for all $X \subseteq A$.

b) G contains a perfect matching if and only if $|A| = |B|$, and $|N(X)| \geq |X|$ holds for all $X \subseteq A$.

1.$^-$ Determine the ν and τ parameters of the following graph:



2. On a 8×8 chessboard we want to draw as many diagonals (of some of the small squares) as we can, such that no two of these diagonal has a common point. (No common endpoints are allowed neither.) At most how many such diagonals can be drawn?

3. G is a simple bipartite graph with partite classes A and B of the same size. Prove that if there is no isolated vertex in G and the degrees of vertices of A are all different, then G contains a perfect matching.

4. We have n bins and n objects. Each object fits into all but at most one bins. Show that all the objects can be packed into these bins.

5. Prove that every d-regular bipartite graph contains a perfect matching, if $d \geq 1$.

6. There are n clubs in the class. Every club has 4 members, and every student is the member of precisely 3 clubs. Prove that it is possible to elect club leaders so that nobody is the leader of more than one club. (The leader must be a member of its club.)

7. The first r rows of an $n \times n$ array are filled with the numbers $1, \ldots, n$ such that every number appears at most once in every row and column. Prove that the remaining cells of the array can be filled with the numbers $1, \ldots, n$ so that the whole array has the property that every number appears at most once in every row and column.

8. Let n and k be positive integers such that $n \geq 2k + 1$, and let $\binom{[n]}{k}$ denote the set of k-element subsets of $\{1, 2, \ldots, n\}$. Prove that there exists an *injective* mapping ϕ: $\binom{[n]}{k} \to \binom{[n]}{k+1}$ such that $\phi(H) \supset H$ holds for all $H \in \binom{[n]}{k}$.

9. G is a simple bipartite graph with partite classes A and B, where $|A| = |B| = m$. Assume that every vertex of G has degree at least $m/2$. Prove that G contains a perfect matching.

10. An island of has n married hunter/farmer couples. The Ministry of Hunting divides the island into n equal-sized hunting regions. The Ministry of Agriculture divides it into n equal-sized farming regions. However, the Ministry of Marriage requires that each couple receives two overlapping regions. Prove that it is always possible to assign to each couple one hunting region and one farming region in such a way that they overlap.

11. A doubly stochastic matrix Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that in a doubly stochastic matrix Q, there exists n *non-zero* entries such that each row and each column contains exactly one of these entries.

12.$^+$ Let G be a simple bipartite graph with partite classes A and B such that every vertex of A has odd degree. Moreover, any two different vertices of A has an even number of common neighbors. Show that there is a matching in G which covers A.

13.$^+$ Prove that the edges of a $2k$-regular graph can be partitioned into k classes such that the edges in each class form a 2-regular spanning subgraph. [7.40]

14.$^{+}$ Two people perform a card trick. The first performer takes 5 cards from a 52-card deck (previously shuffled by a member of the audience), looks at them, keeps one of the cards and arranges the remaining four in a row from left to right, face up. The second performer guesses correctly the hidden card.

a) Prove that the performers can agree on a system which always makes this possible.

b) Devise one such system.