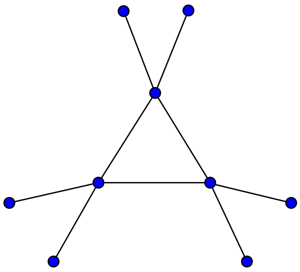
## 2. Enumeration of spanning trees

**1.** How many spanning trees does the following graph have?



**2.** Among the spanning trees of $K_n$, how many stars and paths are (if $n \geq 3$)?

**3.** Replace every edge of the complete graph $K_n$ by two parallel edges. How many spanning trees does the obtained graph have?

**4.** Count the number of spanning of $K_n$ in which a fixed vertex $v$ is a leaf. (Here $n \geq 3$.)

**5.** What is the number of all trees on $n$ vertices with exactly $n - l$ leaves? [4.8]

**6.** Let $t(G)$ denote the number of spanning trees of $G$, and let $e$ be an arbitrary edge in the graph $G$. Prove that

$$t(G) = t(G - e) + t(G/e),$$

where the $G - e$ is the graph obtained by removing $e$ from $G$, and $G/e$ is the graph obtained from $G$ by contracting $e$.

**7.** What is the number of trees on vertex set $\{v_1, \dots v_n, w_1, \dots, w_m\}$ in which each edge joins a $v_i$ to a $w_j$? [4.11]

**8.** We remove an edge from the complete graph on $n$ vertices ($n \geq 3$). How many spanning trees does the obtained graph have?

**9.**$^+$ Let $n \geq 2$ be an integer. We connect every vertex of an $n$-vertex path to a new external vertex. Show that the obtained $(n+1)$-vertex graph has $F_{2n-1}$ spanning trees, where $F_{2n-1}$ is a Fibonacci number (with indexing $F_0 = F_1 = 1$, $F_2 = 2$, $\dots$ ).

**10.**$^+$ $n$ different colors are given: $c_1, \dots, c_n$. In how many ways a circle of color $c_1$, a circle of color $c_2$, … , and a circle of color $c_n$ can be placed in the plane, so that these circles are pairwise non-intersecting? (Two placements are considered the same if and only if one placement can be continuously transformed into the other one by moving and scaling the circles such that the non-intersecting property of circles is retained during the transformation.)

**11.**$^+$ In a simple *bipartite* graph $G$, every vertex has even degree. Prove that $G$ has an even number of spanning trees.

*Note.* No elementary solution to this problem is known.

**12.**$^+$ Prove Abel's identity combinatorially:

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}.$$

**13.** Count the number of full binary trees with $n + 1$ leaves.

**14.** Count the number of plane trees with $n$ edges.