## 8. Extremal graph theory

*In the following exercises, all graphs are simple!*

**1.**$^-$ $G$ is a simple graph on $n$ vertices (where $n \geq 4$) such that any two edges of $G$ share a common endpoint. Show that $G$ has at most $n-1$ edges.

**2.** At most how many edges $G$ can have, if $G$ is simple graph on $n$ vertices that does not contain
a)$^-$ a cycle;
b) odd cycle;
c) triangle;
d)$^+$ even cycle? [10.1]

**3.** How many edges an $n$-vertex simple graph $G$ can have, if its chromatic number is $k$?

**4.** Let $V$ denote a finite set of points on a unit circle $S^1$ in the plane. By a unit circle we understand the set $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.
a) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.8$?
b) What is the maximum number of unordered pairs $u, v \in V$ such that $|uv| \geq 1.5$?

**5.** Prove that if the simple graph $G$ has $n$ vertices and no cycles of length shorter than $2k+1$, then the number of edges in $G$ is less than $n(n^{1/k} + 1)$.

**6.** (Kővári–Sós–Turán theorem.) The simple graph $G$ has no subgraph $K_{r,s}$, where $2 \leq r \leq s$. Prove that $G$ has at most $\frac{\sqrt[r]{s-1}}{2} n^{2-1/r} + \frac{r-1}{2} n = O(n^{2-1/r})$ edges. [10.37]

**7.** Let $S$ be a set of $n$ points in the plane, where $n \geq 3$ and the distance between any two points of $S$ is at least one. Show that no more than $3n-6$ pairs of points of $S$ can be at distance exactly one.

**8.** Show that if the chromatic number of $G$ is $k$, then $G$ contains all trees on $k$ vertices (as subgraphs).