2. Enumeration of spanning trees

1.$^-$ Give the number of spanning trees of the graph
a) P_n, the path with n edges;
b) C_n, the cycle on n vertices.

2.$^-$ Among the spanning trees of K_n, how many stars and paths are (if $n \geq 3$)?

3.$^-$ Count the number of spanning of K_n in which a fixed vertex v is a leaf. (Here $n \geq 2$.)

4. What is the number of all trees on n vertices with exactly $n - l$ leaves? [4.8]

5. Let $t(G)$ denote the number of spanning trees of G, and let e be an arbitrary edge in the graph G. Prove that

$$t(G) = t(G - e) + t(G/e),$$

where the $G - e$ is the graph obtained by removing e from G, and G/e is the graph obtained from G by contracting e.

6. What is the number of trees on vertex set $\{v_1, \dots v_n, w_1, \dots, w_m\}$ in which each edge joins a v_i to a w_j? [4.11]

7. We remove an edge from the complete graph on n vertices ($n \geq 3$). How many spanning trees does the obtained graph have?

8.$^+$ Let $n \geq 2$ be an integer. We connect every vertex of an n-vertex path to a new external vertex. Show that the obtained $(n+1)$-vertex graph has F_{2n-1} spanning trees, where F_{2n-1} is a Fibonacci number (with indexing $F_0 = F_1 = 1$, $F_2 = 2, \dots$).

9.$^+$ n different colors are given: $c_1, \dots, c_n$. In how many ways a circle of color c_1, a circle of color c_2, ... , and a circle of color c_n can be placed in the plane, so that these circles are pairwise non-intersecting? (Two placements are considered the same if and only if one placement can be continuously transformed into the other one by moving and scaling the circles such that the non-intersecting property of circles is retained during the transformation.)

10.$^+$ Prove Abel's identity combinatorially:

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}.$$

11. Count the number of complete planar binary trees with $n+1$ leaves.

12. Count the number of planar trees with n edges.