1. Degrees

1. $^{-}$ Does there exist a graph with degree sequence
a) $9, 7, 6, 6, 5, 4, 3, 3, 3, 1$
b) $8, 7, 6, 6, 5, 4, 3, 3, 3, 1$?

2. (Havel–Hakimi algorithm.) Using the Havel–Hakimi algorithm, decide whether there exists a *simple* graph with degree sequence
a) $7, 4, 3, 3, 3, 3, 2, 1, 0$
b) $8, 8, 6, 6, 6, 5, 3, 2, 2$.

3. How many simple graphs have degree sequence $7, 7, 7, 7, 5, 5, 4, 4$ (up to isomorphism)?

4. Does there exist a *connected* graph with degree sequence $4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1$?

5. Does there exist a *bipartite* graph with degree sequence $9, 9, 9, 9, 6, 6, 6, 5, 3, 3, 3, 3, 3, 3, 3$?

6. Prove that a k-regular *simple* graph on n vertices exists if and only if kn is even and $k \le n-1$. [5.2]

7. Show that a graph G on $2n$ vertices is regular if and only if for all $A \dot\cup B = V(G)$ vertex partition with $|A| = |B|$, the induced subgraphs $G|_A$ and $G|_B$ have the same number of edges.

8. The **Erdős–Gallai theorem** states that the sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ of nonnegative integers can be realized by a simple graph if and only if

$$d_1 + \cdots + d_n \text{ is even,} \quad \text{and} \tag{1}$$

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \quad \text{for all } k \in \{1, \ldots, n\}. \tag{2}$$

Prove that the conditions are necessary.

9. $^{-}$ Show that if every vertex has degree at least 2 in a simple graph G, then G contains a cycle of length at least $\delta(G) + 1$, where $\delta(G)$ denotes the minimum degree in G.

10. Show that if every vertex has degree at least 3 in a simple graph G, then G contains a cycle of even length. [10.2]

11. $^{+}$ Show that if every vertex has degree at least 3 in a simple graph G, then G contains a subdivision of K_4. [10.3]

12. Let $n \ge 2$ be an integer, and assume that the sequence $d_1, \ldots, d_n$ can be realized by a simple graph. Prove that this sequence can be realized by a *connected* simple graph, if and only if $\sum_{i=1}^{n} d_i \ge 2(n-1)$ and the elements $d_1, \ldots, d_n$ are all positive.

13. Twelve dwarves live in the woods in red or blue houses. In the i'th month of the year the i'th dwarf visits all his friends to decide about repainting his house ($i = 1, \ldots, 12$). He will repaint his house (from blue to red, or vice versa) if and only if the (strict) majority of his friends live in a house of different color. This happens every year. Prove that after a while, noone will repaint his house anymore. (The friendships are mutual and don't change as time passes. Supposedly, not everyone is everyone's friend.)

14. $^{+}$ An arbitrary loopless graph G is given. Prove that it is possible to remove some edges of G so that the obtained graph G' is bipartite, and in G' each vertex has degree at least half of its original degree (in G).
Corollary (BSc): Every loopless graph can be made bipartite by the deletion of at most half of its edges.

15. In a graph G the average degree is $\bar{d}(G)$. Prove that there is an induced subgraph S in G for which $\delta(S) \geq \frac{\bar{d}(G)}{2}$, where $\delta(S)$ is the minimum degree in the graph S.

16.$^{+}$ Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of $t_n + 1$ people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following conditions to hold at the same time:

(i) The number of games played by each person is one of $t_1, t_2, \ldots, t_n$;

(ii) For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

17.$^{+}$ An arbitrary simple graph G and a positive integer k are given. Prove that the vertices of G can be colored red and blue so that every red vertex has less than k red neighbours, and every blue vertex has at least k red neighbours.