Linear algebra in photonics

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1 Introduction, review

Exercise 1 *Plot the following complex numbers in the complex plane:* 1, i, $-i$, $3+2i$, $-3+4i$.

Exercise 2 *Do the following calculations.*

a) $i + (2 - 3i)$

b) i^2

c) i^3

d) $(2 + 3i) + (3 - 4i)$

e) $(2 + 3i) \cdot (3 - 4i)$

Exercise 3 *Do the following calculations. Write out the real and imaginary parts.*

a) $\overline{2 - 3i}$

b) $\overline{(4 + 7i) + (3 - 4i)}$

c) $\overline{(2 + 5i)} + \overline{(1 - 4i)}$

d) $\overline{(-1 + 2i) \cdot (3 - i)}$

e) $\overline{(3 + 3i)} \cdot \overline{(1 - 4i)}$

Exercise 4 *Calculate the absolute value of the following complex numbers:* $3 + 4i$, $-12 + 5i$, $-7 - 24i$.

Exercise 5 *Do the following calculations. We expect the results in canonical (Cartesian) form.*

a) $\frac{1+i}{i}$

b) $\frac{2}{4-3i}$

c) $\frac{1+2i}{5-12i}$

Exercise 6 *Justify with calculation that*

$$\frac{\overline{2-4i}}{\overline{2+2i}} = \overline{\left(\frac{2-4i}{2+2i}\right)}.$$

Exercise 7 *Justify with calculation that*

a) $|2+3i| \cdot |1-i| = |(2+3i)\cdot(1-i)|$.

b) $\frac{|2-4i|}{|2+2i|} = \left|\frac{2-4i}{2+2i}\right|$.

Exercise 8 *Find the real numbers a and b such that*

a) $(a+bi)(2-i) = a+3i$.

b) $(a+i)(1+bi) = 3b+ai$.

Exercise 9 *Find the trigonometric form of the following complex numbers:* -3, $4i$, $1-i$, $\sqrt{3}+i$.

Exercise 10 *Do the following calculations. We expect the results in both canonical and trigonometric forms.*

a) $(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

b) $[3(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})]^2$

c) $\frac{4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})}$

Exercise 11 *Using the trigonometric form do the following calculation.*

$$\frac{(1+i)^9}{(1-i)^7}$$

Exercise 12 *Find the square roots of the following complex numbers.*

a) 4.

b) -9.

c) $-7-24i$.

Exercise 13 *Solve the equation* $(4+i)z+3-i=0$ *on the set of complex numbers.*

Exercise 14 *Solve the equation* $z^2+2z+5=0$ *on the set of complex numbers.*

Exercise 15 *Solve the equation* $(2+i)z^2-(5-i)z+(2-2i)=0$ *on the set of complex numbers.*

Exercise 16 *Solve the equation* $z^2-(3-2i)z+(5-5i)=0$ *on the set of complex numbers.*

Exercise 17 *Find all the complex numbers whose 7th power is* 1. *Use the trigonometric form.*

Exercise 18 *Find the 3rd roots of* $z=2+2i$.

Exercise 19 *Find 5th roots of* $z=-16\sqrt{3}+16i$.

Exercise 20 *Solve the following equation.*

$$z^4 - \frac{7+3i}{5-2i} = 0$$

Exercise 21 *Compute the scalar product of* $\mathbf{u}(-5,1,-5)$ *and* $\mathbf{v}(-1,0,-5)$.

Exercise 22 *Compute the scalar product of* $\mathbf{u}(t+3, -4t+2, -2t-5)$ *and* $\mathbf{v}(1, -1, 5)$.

Exercise 23 *Find* t *such that* $\mathbf{u}(-5t+5, 2t-2, t+5)$ *and* $\mathbf{v}(4, 5, -1)$ *are perpendicular.*

Exercise 24 *Given the vectors* $\mathbf{u}(5, -3, 4)$ *and* $\mathbf{v}(5, -2, -4)$, *find* t *such that the vectors* $\mathbf{u}$ *and* $t\mathbf{u} + \mathbf{v}$ *are perpendicular.*

Exercise 25 *Compute the length of* $\mathbf{v}(-3, -1, 3)$.

Exercise 26 *Compute the angle between the vectors* $\mathbf{u}(-3, 3, -2)$ *and* $\mathbf{v}(-3, -1, 0)$.

Exercise 27 *Compute the vectorial product of* $\mathbf{u}(0, -4, -4)$ *and* $\mathbf{v}(1, 3, 1)$.

Exercise 28 *Compute the vectorial product of* $\mathbf{u}(t-4, 4t+3, -2t+2)$ *and* $\mathbf{v}(-4, 1, 1)$.

Exercise 29 *Compute the* $\|\cdot\|_1$, $\|\cdot\|_2$ *and* $\|\cdot\|_\infty$ *norms of the vectors* $\mathbf{v}(-8, 5, 2)$ *and* $\mathbf{w}(1, 0, 1)$.

Exercise 30 *Compute the* $\|\cdot\|_1$, $\|\cdot\|_2$ *and* $\|\cdot\|_\infty$ *norms of the vectors* $\mathbf{v}(-3, 3, 1)$ *and* $\mathbf{w}(1, 4, 2)$.

Exercise 31 *Find the product* AB *of the following matrices.*

$$A = \begin{pmatrix} -1 & 0 & -4 \\ 0 & 2 & 0 \\ -2 & 4 & -2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -4 & 2 \\ -5 & 2 & 4 \\ -3 & -2 & 2 \end{pmatrix}$$

Exercise 32 *Find the product* AB *of the following unipotent matrices.*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 5 & 0 & -3 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 33 *Compute* $f(A)$ *where* $f(x) = x^2 + 3x - 4$ *and*

$$A = \begin{pmatrix} 1 & 3 & -5 \\ 4 & -2 & 6 \\ 3 & 1 & 2 \end{pmatrix}.$$

Exercise 34 *Find the determinant of the matrix* $A = \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}$.

Exercise 35 *Find the determinant of the lower triangular matrix* $A = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & -2 & -2 \end{pmatrix}$.

Exercise 36 *Find the determinant of the matrix* $A = \begin{pmatrix} -4 & 0 & 5 \\ 1 & -1 & -4 \\ -1 & 2 & -5 \end{pmatrix}$.

Exercise 37 *Find the determinant of the following matrices.*

a)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ -2 & -2 & 11 \end{pmatrix}$$

c)

$$C = \begin{pmatrix} -1 & 3 & -4 \\ -1 & 1 & -2 \\ -2 & 4 & -6 \end{pmatrix}$$

d)

$$D = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 1 & 5 & 2 & 0 \\ 0 & 4 & 2 & -1 \\ 1 & -1 & 0 & 3 \end{pmatrix}$$

Exercise 38 *Find x.*

$$\det \begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & 3 \\ 1 & -1 & x \end{vmatrix} = 8$$

Exercise 39 *Find x.*

$$\det \begin{vmatrix} x & 1 & 2 \\ 2 & 1 & 0 \\ -2 & -1 & x \end{vmatrix} = 4$$

Exercise 40 *Find the $\|\cdot\|_1$ and the $\|\cdot\|_\infty$ norm of the following matrix.*

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ -2 & -2 & 11 \end{pmatrix}$$

Exercise 41 *Find the $\|\cdot\|_1$ and the $\|\cdot\|_\infty$ norm of the following matrix.*

$$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 1 & 5 & 2 & 0 \\ 0 & 4 & 2 & -1 \\ 1 & -1 & 0 & 3 \end{pmatrix}$$

2 Gaussian elimination

Exercise 42 *Solve the following system of linear equations.*

$$\begin{cases} x_1 + x_2 = 1 \\ -x_1 - 3x_2 = -4 \end{cases}$$

Exercise 43 *Back solve the following upper triangular system of linear equations.*

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \quad -3x_2 + x_3 = -2 \\ \quad\quad 2x_3 = 8 \end{cases}$$

Exercise 44 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 = 6 \\ -x_2 - 2x_3 = -2 \\ -x_1 + 2x_3 = 3 \end{cases}$$

Exercise 45 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} x_1 + 2x_2 + 5x_3 = -9 \\ x_1 - x_2 + 3x_3 = 2 \\ 3x_1 - 6x_2 - x_3 = 25 \end{cases}$$

Exercise 46 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} x_1 + 2x_2 - x_3 = -2 \\ x_1 - x_2 + x_3 = 1 \\ x_1 - 3x_2 + 4x_3 = 8 \end{cases}$$

Exercise 47 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} x_2 - 3x_3 + 4x_4 = -5 \\ x_1 - 2x_3 + 3x_4 = -4 \\ 3x_1 + 2x_2 - 5x_4 = 12 \\ 4x_1 + 3x_2 - 5x_3 = 5 \end{cases}$$

Exercise 48 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 8 \\ x_2 + x_3 = 3 \\ x_1 - x_2 + 2x_3 = 3 \end{cases}$$

Exercise 49 *Using Gaussian elimination solve the following system of linear equations.*

$$\begin{cases} x_1 + 2x_2 + x_3 = 8 \\ 2x_1 - x_2 + 3x_3 = 9 \\ x_1 - 3x_2 + 2x_3 = 18 \end{cases}$$

Exercise 50 *Using Gaussian elimination solve the following system of linear equations. (a is a real parameter.)*

$$\begin{cases} x_1 - x_2 + x_3 + ax_4 = 1 \\ x_1 + (1-a)x_3 + (a-1)x_4 = 2 \\ x_1 - ax_3 + (a-2)x_4 = 1 \\ -ax_1 + ax_2 + 2ax_3 + 2x_4 = 3a-1 \end{cases}$$

Exercise 51 *Using Gaussian elimination solve the following system of linear equations. (a is a real parameter.)*

$$\begin{cases} x_1 - 2x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = 3 \\ x_1 - 2x_2 + (a^2-8)x_3 = a+4 \end{cases}$$

Exercise 52 *Find the LU decomposition of the matrix* $A = \begin{pmatrix} 1 & -4 & 3 \\ -4 & 17 & -14 \\ -5 & 21 & -16 \end{pmatrix}$.

Exercise 53 *Find the LU decomposition of the matrix* $A = \begin{pmatrix} 1 & 4 & -5 \\ -3 & -11 & 18 \\ 5 & 20 & -24 \end{pmatrix}$.

Exercise 54 *Find the LU decomposition of the matrix* $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -2 \\ -4 & -6 & 5 \end{pmatrix}$.

Exercise 55 *Find the LU decomposition of the matrix* $A = \begin{pmatrix} 4 & -1 & 4 \\ -4 & 3 & -2 \\ 5 & -1 & 4 \end{pmatrix}$.

Exercise 56 *Find the LU decomposition of the matrix* $A = \begin{pmatrix} 4 & 2 & 2 \\ -3 & 4 & -4 \\ 3 & 1 & -5 \end{pmatrix}$.

Exercise 57 *Solve the following system of lieanr equations*

$$\begin{aligned} x_1 - 4x_2 - x_3 &= -15 \\ -3x_1 + 13x_2 + 2x_3 &= 47 \\ 5x_1 - 18x_2 - 6x_3 &= -70 \end{aligned}$$

where the LU decomposition of the coefficient matrix is given:

$$\begin{pmatrix} 1 & -4 & -1 \\ -3 & 13 & 2 \\ 5 & -18 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 58 *Solve the following system of lieanr equations*

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ 5x_1 + 11x_2 - 3x_3 &= 25 \\ -x_1 - 3x_2 + 4x_3 &= 8 \end{aligned}$$

where the LU decomposition of the coefficient matrix is given:

$$\begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & -3 \\ -1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 59 *Find the inverse of* $A = \begin{pmatrix} -2 & 0 \\ -4 & 2 \end{pmatrix}$.

Exercise 60 *Find the inverse of* $A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$.

Exercise 61 *Find the inverse of the unipotent matrix* $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}$.

Exercise 62 *Find the inverse of the upper triangular matrix* $A = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{pmatrix}$.

Exercise 63 *Find the inverse of* $A = \begin{pmatrix} -4 & -1 & 4 \\ 3 & -4 & 0 \\ 2 & -3 & 0 \end{pmatrix}$.

Exercise 64 *Find the matrix* X *such that*

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \cdot X = \begin{pmatrix} 6 & 0 \\ 10 & 1 \end{pmatrix}.$$

Exercise 65 *Find the matrix* X *such that*

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 0 & -5 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 4 & -1 & 2 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Exercise 66 *Find the matrix* X *such that*

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 0 & -5 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}.$$

Exercise 67 *Find the matrix* X *such that*

$$X \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}.$$

Exercise 68 *Compute the natural cubic interpolant of the points* $(0, 2), (4, 0), (8, 6), (12, 4)$.

Exercise 69 *Compute the natural cubic interpolant of the points* $(-2, 4), (0, 10), (8, 2), (10, 8)$.

3 Least squares problem, Gram-Schmidt method

Exercise 70 *Find the minimum of* $x^2 + y^2$ *provided* $3x - 4y = 5$.

Exercise 71 *Find the minimum of* $x^2 + y^2 + z^2$ *provided* $x - 4y - 2z = 21$.

Exercise 72 *Find the vector* $\mathbf{x} = (x_1, x_2, x_3)^T$ *of minimal* $\|\cdot\|_2$ *norm for which*

$$\begin{cases} x_1 - 2x_2 + x_3 = 1 \\ x_1 - 3x_2 + 2x_3 = -1 \end{cases}$$

Exercise 73 *Find the vector* $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ *of minimal* $\|\cdot\|_2$ *norm for which*

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

Exercise 74 *Find the vector* $\mathbf{x} = (x_1, x_2)^T$ *for which* $\|A\mathbf{x} - \mathbf{b}\|_2$ *is minimal and*

$$A = \begin{pmatrix} -4 & -1 \\ 3 & -4 \\ 2 & 0 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

Exercise 75 *Find the real number* λ *for which* $(2\lambda - 5)^2 + (3\lambda - 1)^2$ *is minimal! Apply various methods, and compare the results*

Exercise 76 *Find the vector* $\mathbf{x} = (x_1, x_2)^T$ *for which* $\|A\mathbf{x} - \mathbf{b}\|_2$ *is minimal andl*

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}.$$

Exercise 77 *Find the vector* $\mathbf{x} = (x_1, x_2, x_3)^T$ *for which* $\|A\mathbf{x} - \mathbf{b}\|_2$ *is minimal and*

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 4 & 3 & 2 \\ 0 & -2 & 1 \\ 5 & 4 & 0 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 78 *Find the fitting line of the points* $(1, -1)$*,* $(2, 3)$ *and* $(4, 6)$*.*

Exercise 79 *Find the fitting line of the points* $(0, -1)$*,* $(1, 1)$*,* $(2, 3)$*,* $(2, 4)$ *and* $(3, 6)$*.*

Exercise 80 *Find the fitting plane of the points* $(0, -1, 1)$*,* $(1, -1, 2)$*,* $(0, 2, 3)$*, and* $(3, 6, 1)$*.*

Exercise 81 *Find two orthogonal vectors that span the same plane as the vectors* $(1, 2, 3)$ *and* $(1, 0, -1)$*.*

Exercise 82 *Find an orthogonal system of vectors in* $\mathbb{R}^4$ *that span the same linear subspace as* $(1, 0, 2, 4)$*,*$(1, 1, 1, 2)$ *and* $(0, 1, 0, -1)$*.*

Exercise 83 *Find the QR decomposition of the matrix*

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 4 & 3 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

Exercise 84 *Find the QR decomposition of the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ -2 & 1 & 2 & 1 \\ 0 & 2 & 3 & -1 \end{pmatrix}$$

4 Eigenvalue, eigenvector

Exercise 85 *Verify that* 2 *is an eigenvalue of* A*, and find the corresponding eigenvectors.*

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 8 & 1 \\ 1 & 6 & 3 \end{pmatrix}$$

Exercise 86 *Compute the characteristic polynomial of the following matrices.*

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 8 & 1 \\ 1 & 6 & 3 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & \pi & e \\ 0 & 1 & x^3 \\ 0 & 0 & 3 \end{pmatrix} \quad , \quad C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exercise 87 *Find the eigenvalues and corresponding eigenvectors of the following matrices.*

a)

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

c)

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

d)

$$D = \begin{pmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

e)

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 88 *Find the eigenvalues of the matrix* $B = \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix}$.

Exercise 89 *Find the eigenvalues of the matrix* $B = \begin{pmatrix} 6 & 0 \\ 6 & 3 \end{pmatrix}$.

Exercise 90 *Find the eigenvalues and the corresponding eigenvectors of the matrix* $B = \begin{pmatrix} 2 & -4 \\ 5 & -7 \end{pmatrix}$.

Exercise 91 *Find the eigenvalues and the corresponding eigenvectors of the matrix* $B = \begin{pmatrix} -6 & -6 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{pmatrix}$.

Exercise 92 *Find the eigenvalues and the corresponding eigenvectors of the matrix* $B = \begin{pmatrix} 2 & -3 & -3 \\ 25 & 86 & 59 \\ -37 & -123 & -84 \end{pmatrix}$.

Exercise 93 *Compute the* $\|\cdot\|_1, \|\cdot\|_2$ *and* $\|\cdot\|_\infty$ *norms of the matrix*

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}.$$

Exercise 94 *Compute the* $\|\cdot\|_1, \|\cdot\|_2$ *and* $\|\cdot\|_\infty$ *norms of the matrix*

$$\begin{pmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ -2 & 0 & 3 \end{pmatrix}.$$

Exercise 95 *Find the matrices* A *and* D*, wherel* D *is diagonal, and for* $B = \begin{pmatrix} -3 & 4 \\ -6 & 7 \end{pmatrix}$ *we have* $B = ADA^{-1}$.

Exercise 96 *Compute*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{2014}.$$

Exercise 97 *Compute*

$$\begin{pmatrix} -3 & 4 \\ -6 & 7 \end{pmatrix}^{2014}.$$

Exercise 98 *Consider the symmetric matrix*

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$

and apply one Jacobi iteration to obtain the eigenvalues of A.

Exercise 99 *Consider the symmetric matrix*

$$B = \begin{pmatrix} -6 & 4 & -3 \\ 4 & 2 & 0 \\ -3 & 0 & -3 \end{pmatrix},$$

and apply one Jacobi iteration to knock out the element $b_{13} = -3$.

Exercise 100 *Consider the symmetric matrix*

$$C = \begin{pmatrix} -3 & 2 & -3 \\ 2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix},$$

and using the Jacobi method find its eigenvalues with two digits accuracy.

Exercise 101 *Using that*

$$A = \begin{pmatrix} 1.8 & -2.4 \\ 2.4 & -3.2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix},$$

apply one QR *iteration to* A*, and find its eigenvalues.*

Exercise 102 *Consider the matrix*

$$A = \begin{pmatrix} 6 & 2 \\ 8 & 11 \end{pmatrix}.$$

Using the characteristic polynomial compute the eigenvalues of A*. Then apply two* QR *iterations to* A*, end compute the difference between the eigenvalues and the elements in the main diagonal.*

5 Jacobi iteration

Exercise 103 *Start with initial values* $(1, 1, 1)$*, and apply three Jacobi iterations to estimate the roots of the following system of linear equations. What do we obtain?*

$$\begin{cases} 15x_1 + x_2 + 2x_3 = 3 \\ 10x_2 + 3x_3 = -6 \\ x_3 = 18 \end{cases}$$

Exercise 104 *Start with initial values* $(1, 2, 3)$*, and apply three Jacobi iterations to estimate the roots of the following system of linear equations. What do we obtain?*

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 13 \\ x_2 + x_3 = 4 \\ 2x_3 = 8 \end{cases}$$

Exercise 105 *Start with initial values* $(1, 1, 1)$*, and apply three Jacobi iterations to estimate the roots of the following system of linear equations.*

$$\begin{cases} 6x_1 + 2x_2 + x_3 = 8 \\ x_1 - 5x_2 + 3x_3 = 9 \\ x_1 - x_2 + 3x_3 = 18 \end{cases}$$

Exercise 106 *Start with initial values* $(1, 1, 1)$*, and apply three Jacobi iterations to estimate the roots of the following system of linear equations.*

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 15 \\ x_1 - x_2 + 3x_3 = 4 \\ 2x_1 - x_2 + 3x_3 = 1 \end{cases}$$

6 Linear programming

Exercise 107 *Solve the following LP problem via the graphical method.*

$$\begin{array}{ccc} x_1 + x_2 & \leq & 20 \\ x_1 & \leq & 10 \\ x_1, x_2 & \geq & 0 \\ \hline 6x_1 + 2x_2 & \to & \text{max!} \end{array}$$

Exercise 108 *Solve the following LP problem via the graphical method.*

$$\begin{array}{ccc} x_1 + x_2 & \leq & 20 \\ x_1 & \leq & 10 \\ x_1, x_2 & \geq & 0 \\ \hline 2x_1 + 6x_2 & \to & \text{min!} \end{array}$$

Exercise 109 *Solve the following LP problem via the graphical method.*

$$\begin{array}{ccc} x_1 + x_2 & \leq & 20 \\ x_1 & \leq & 10 \\ x_1, x_2 & \geq & 0 \\ \hline 5x_1 + 3x_2 & \to & \text{max!} \end{array}$$

Exercise 110 *Solve the following LP problem via the graphical method.*

$$\begin{array}{rcl} x_1 + 2x_2 & \leq & 25 \\ x_1 + 3x_2 & \leq & 33 \\ x_1 + x_2 & \leq & 20 \\ x_1, x_2 & \geq & 0 \\ \hline 3x_1 + 5x_2 & \to & \max! \end{array}$$

Exercise 111 *Solve the following LP problem via the graphical method.*

$$\begin{array}{rcl} 8x_1 + 8x_2 & \leq & 64 \\ x_1 + 3x_2 & \leq & 15 \\ 3x_1 & \leq & 18 \\ 2x_2 & \leq & 8 \\ x_1, x_2 & \geq & 0 \\ \hline 5x_1 + 3x_2 & \to & \max! \end{array}$$

Exercise 112 *Solve the following LP problem via the simplex method.*

$$\begin{array}{rcl} x_1 + x_3 & \leq & 40 \\ -x_2 + x_3 & \leq & 10 \\ x_1 + x_2 - x_3 & \leq & 18 \\ x_1, x_2, x_3 & \geq & 0 \\ \hline 4x_1 + 3x_3 & \to & \max! \end{array}$$

Exercise 113 *Solve the following LP problem via the simplex method.*

$$\begin{array}{rcl} x_1 + 2x_2 + x_3 + x_5 & \leq & 100 \\ x_2 + x_3 + x_4 + x_5 & \leq & 80 \\ x_1 + x_3 + x_4 & \leq & 50 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \\ \hline 2x_1 + x_2 + 3x_3 + x_4 + 2x_5 & \to & \max! \end{array}$$

Exercise 114 *Solve the following LP problem via the simplex method.*

$$\begin{array}{rcl} x_1 - 2x_2 & \leq & 6 \\ -x_1 - 3x_3 & = & -7 \\ x_1 - x_2 - 3x_3 & \leq & -2 \\ 3x_1 - 2x_2 - 2x_3 & \leq & 4 \\ x_1, x_2, x_3 & \geq & 0 \\ \hline 4x_1 - 2x_2 + x_3 & \to & \max! \end{array}$$

7 Discrete Fourier Transform

Exercise 115 *Compute the DFT of the vector* $\mathbf{v} = (2, 4, 8, 16)^T$.

Exercise 116 *Compute the DFT of the vector* $\mathbf{v} = (2i, 2 + i, i, 1 - i)^T$.

Exercise 117 *Compute the inverse DFT of the vector* $\mathbf{v} = (2, 4, 8, 16)^T$.

Exercise 118 *Compute the inverse DFT of the vector* $\mathbf{v} = (2i, 2 + i, i, 1 - i)^T$.

Exercise 119 *Compute the DFT of the vector* $\mathbf{v} = (2, 4, 8, 16, 32, 64, 128, 256)^T$ *using the Fast Fourier Transform algorithm.*

Exercise 120 *Compute the DFT of the vector* $\mathbf{v} = (1 + 3i, 2, 2 - i, 6 + 4i, 2i, 4, 3 - 3i, i)^T$ *using the Fast Fourier Transform algorithm.*