BLOCKING s-SPACES BY t-SPACES IN $\mathbb{F}_q^n$

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Let $0 \leqslant t \leqslant s \leqslant t \leqslant n$ be integers. In the n-dimensional vector space $\mathbb{F}_q^n$ over the q element field, an (s,t)-*blocking set* is a set t-spaces such that each s-space is incident with at least one chosen t-space. Denote by $f_{s,t}(n,q)$ the cardinality of the smallest such a blocking set. It is a trivial folklore result that $f_{s,t}(n,q) = q^N + O(q^{N-1})$ as $q \to \infty$ for $N \coloneqq (n-s)t$, but determining $f_{s,t}(n,q)$ more precisely is a notoriously difficult problem, as it is equivalent to determining the size of certain q-Turán designs and q-covering designs. For example, the exact value of even $f_{3,2}(n,q)$ is known only for $n \leqslant 5$.

We present an improvement on the upper bounds of Eisfeld and Metsch [3, Theorem 1.2.], [2, Theorem 1.2] for $(s,t) = (3,2)$ via a refined scheme for a recursive construction, which in fact enables improvement in the general case as well.

Theorem 1 ([1, Theorem 1.9])**.** *Let $(s,t) = (3,2)$ and $n \geqslant 6$. Then as $q \to \infty$, we have*

$$f_{3,2}(n,q) \leqslant \mathbf{1}q^N + \mathbf{0}q^{N-1} + \mathbf{2}q^{N-2} + 2q^{N-3} + 3q^{N-4} + 3q^{N-5} + 3q^{N-6} + 3q^{N-7} + O(q^{N-8}),$$
$$f_{3,2}(n,q) \geqslant \mathbf{1}q^N + \mathbf{0}q^{N-1} + \mathbf{2}q^{N-2} + q^{N-3} + 2q^{N-4}.$$

Theorem 2 (General recursive construction, [1, Corollary 3.7])**.** *Let X be an n-dimensional vector space, $K \leqslant X$ be a k-space with $k \leqslant n-s$. For each integer $0 \leqslant i \leqslant \min\{k,s\}$, pick an arbitrary integer $t_i \in [0,i] \cap [t-(s-i),t] \neq \varnothing$.*

If $\mathcal{B}_K(i)$ is an (i,t_i)-blocking set in K, and $\mathcal{B}_Q(i)$ is an $(s-i,t-t_i)$-blocking set in $Q \coloneqq X/K$, then

$$\mathcal{B}_X \coloneqq \bigsqcup_{i=0}^{\min\{k,s\}} \mathcal{B}_K(i) * \mathcal{B}_Q(i)$$

is an (s,t)-blocking in X where $\mathcal{B}_K * \mathcal{B}_Q \coloneqq \{T \leqslant X : K \cap T \in \mathcal{B}_K, \langle K,T\rangle \in \mathcal{B}_Q\}$.

These results are joint work with **Benedek Kovács** and **Zoltán Lóránt Nagy**.

References

[1] Kovács, B., Nagy, Z.L. and Szabó, D.R. *Blocking planes by lines in* PG(n,q), Des. Codes Cryptogr. (2025). `https://doi.org/10.1007/s10623-025-01678-w`

[2] Metsch, K. (2004). *Blocking subspaces by lines in* PG(n,q), Combinatorica **24** 459-486.

[3] Eisfeld, J., Metsch, K. (1997). *Blocking s-dimensional subspaces by lines in* PG$(2s,q)$, Combinatorica, 17(2), 151-162.