On the (weakly) uniform structure of bipartite graphs which admit a dual adjacency matrix (candidate)

Giusy Monzillo

University of Primorska (Slovenia)

The Q-polynomial property of distance-regular graphs was introduced by Delsarte in his doctoral thesis, and it has been extensively studied since then. It is known that if a distance-regular graph is Q-polynomial, then for each vertex x there exists a so-called *dual adjacency matrix with respect to* x, say $A^* = A^*(x)$. Furthermore, in such a case, the adjacency matrix A of the graph and A^* satisfy

$$A^3A^* - A^*A^3 + (\beta+1)(AA^*A^2 - A^2A^*A) = \gamma(A^2A^* - A^*A^2) + \rho(AA^* - A^*A) \qquad (1)$$

for some scalars β, γ, ρ.

In [2], Terwilliger introduced a generalization of the Q-polynomial property: a graph is said to be *Q-polynomial with respect to a vertex x* if it has a *dual adjacency matrix with respect to x*.

The aim of finding examples of graphs with the above *new* property justifies our following definition. Let Γ denote a finite, simple, connected graph with vertex set X. Fix $x \in X$ and let $\varepsilon \geqslant 3$ be the eccentricity of x. For mutually distinct scalars $\{\theta_i^*\}_{i=0}^{\varepsilon}$, define a diagonal matrix $A^* = A^*(\theta_0^*, \theta_1^*, \ldots, \theta_\varepsilon^*) \in \mathrm{Mat}_X(\mathbb{R})$ as follows:

$$(A^*)_{yy} = \theta^*_{\partial(x,y)},$$

where $y \in X$ and ∂ is the shortest path-length distance function of Γ. We say that A^* is a *dual adjacency matrix candidate of Γ with respect to x* if the adjacency matrix $A \in \mathrm{Mat}_X(\mathbb{R})$ of Γ and A^* satisfy (1) for some scalars $\beta, \gamma, \rho \in \mathbb{R}$.

In this talk, we investigate the relation between two *objects* that a bipartite graph can possess: a dual adjacency matrix candidate and a uniform structure (in the sense of Terwilliger [1]). To do that, we first define a *weakly uniform structure* by slightly relaxing the conditions of a uniform structure. The main result is the following:

Theorem ([3]). *A bipartite graph Γ admits a dual adjacency matrix candidate with respect to x if and only if Γ admits a weakly uniform structure with respect to x; in particular, for $\beta = 2$, the latter weakly uniform structure is an actual uniform structure.*

References

[1] P. Terwilliger. The incidence algebra of a uniform poset. *Coding theory and design theory*, Part I, IMA Vol. Math. Appl., **20**, 193-212 (1990).

[2] P. Terwilliger. A Q-Polynomial Structure Associated with the Projective Geometry $L_N(q)$. *Graphs and Combinatorics*, **39**(4): 63, (2023).

[3] B. Fernández, R. Maleki, Š. Miklavič, and G. Monzillo. *On the uniform structure of bipartite graphs admitting a dual adjacency matrix candidate*, preprint.