On finite florets in Hilbert's (geo-)garden

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Finite projective geometries constitute a well-established research topic, with applications in many different areas. However, it could be said that (arguably) the most "basic" geometry is Euclidean geometry. It is founded upon the axiom set established by David Hilbert at the end of the 19^{th} century, consisting of five groups of axioms, the first of which (named "incidence axioms") introduces the primitive notions: points, lines and planes, and the primitive relation called incidence (hence the naming). Incidence axioms allow some finite models, but in stark contrast to projective geometries, in the literature there are practically no results on such models (more or less the only thing that is mentioned here and there is that the minimum number of points needed to model these axioms is 4); a very recent exception is an article [1] from 2024, where the complete catalog of such finite models with up to 12 points was exhibited. In that article, a connection between such finite models and projective planes and spaces, combinatorial designs, as well as matroid theory, has been brought to light.

How can we push the present frontier of this enumeration further? Unfortunately, 12 points is indeed an unbreakable limit for the approach employed in [1]. To overcome this barrier, we discuss the creation of a new axiom system—which is supposed to be equivalent to Hilbert's incidence axioms, but crafted in such a way as to make it amenable to attack by specialized solvers for Boolean satisfiability problems (SAT in short). Because, although the SAT problem is hard in *theory* (which means that a polynomial algorithm does not exist, unless P = NP), in *practice* the situation is not so dire. Namely, various heuristic SAT algorithms exist that perform quite well in problems that arise from practice (either from real-world or from research in other areas), even if they involve thousands or more of variables and/or logical constraints. The idea is to use such state-of-the-art algorithms on our newly constructed axiomatic system, and thereby solve our enumeration problem for larger models.

Along the way, we aim to construct several new families of finite models—which gives a lower bound for the number of models with n points (without any caps on n), but not less importantly, also showcases just how varied these "mini-Euclidean worlds" can be. In doing so, we hope to place finite incidence geometries more firmly on the research map—a century and a quarter after Hilbert first laid down the rules of the game.

References

[1] K. Ago, B. Bašić, M. Maksimović, M. Šobot, On finite models of Hilbert's incidence geometry, *Discrete Math.* **347** (2024), Article No. 114159, 15 pp.