On the set of directions determined by $\mathbb{F}_q$-subspaces of $\mathrm{AG}(r, q^n)$

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(Joint work with Giuseppe Marino and Valentina Pepe)

Consider $\mathrm{PG}(r, q^n)$ as $\mathrm{AG}(r, q^n) \cup H_\infty$, where H_∞ is the hyperplane at infinity. The set of directions determined by an affine point set S of $\mathrm{AG}(n, q^n)$ is the set of ideal points $\mathrm{dir}(S) = \{\langle P, Q\rangle \cap H_\infty : P, Q \in S, P \neq Q\}$. For a function $f\colon \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ the graph of f is the affine point set $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\} \subseteq \mathrm{AG}(2, q^n)$. Assume that f (and hence U_f) is $\mathbb{F}_q$-linear. If ℓ is a line through the origin, then $|U_f \cap \ell| = q^i$, $i \in \mathbb{Z}_0^+$. Let w be maximal such that for each line ℓ through the origin either $U_f \cap \ell = \{(0, 0)\}$, or $|U_f \cap \ell| \geqslant q^w$. By a result of Ball et al. $\mathbb{F}_{q^w}$ is the largest subfield of $\mathbb{F}_{q^n}$ such that U_f is an $\mathbb{F}_{q^w}$-subspace, [1, 2]. With G. Marino and V. Pepe we proved the following generalisation.

Theorem 1. *Let U denote an m-dimensional $\mathbb{F}_q$-subspace of $\mathrm{AG}(r, q^n)$. Let w be maximal such that for each line ℓ through the origin either $U \cap \ell = \{(0, \ldots, 0)\}$ or $|U \cap L| \geqslant q^w$.*

(a) If $n \mid m$, then $w \mid n$ and $\mathbb{F}_{q^w}$ is the largest subfield of $\mathbb{F}_{q^n}$ such that U is $\mathbb{F}_{q^w}$-linear.

If $n \nmid m$ then U is not necessarily linear over a larger field, but it acts similarly:

(b) If $q \geqslant n$, then there exists an integer $d \geqslant w$, $d \mid n$, such that $\mathrm{dir}(U) = \mathrm{dir}(\langle U\rangle_{\mathbb{F}_{q^d}})$.

References

[1] S. Ball: The number of directions determined by a function over a finite field, *J. Combin. Theory Ser. A* **104** (2003), 341–350.

[2] S. Ball, A. Blokhuis, A.E. Brouwer, L. Storme and T. Szőnyi: On the number of slopes of the graph of a function defined over a finite field, *J. Combin. Theory Ser. A* **86** (1999), 187–196.

[3] B. Csajbók, G. Marino, V. Pepe: On the maximum field of linearity of linear sets, submitted manuscript, `https://arxiv.org/abs/2306.07488`.