Projective norm graphs

Lajos Rónyai
BME, SzTAKI

Let $q > 1$ be a prime power, $t > 1$ an integer and denote by $N : \mathbb{F}_{q^{t-1}} \to \mathbb{F}_q$ the norm map from the finite field $\mathbb{F}_{q^{t-1}}$ to $\mathbb{F}_q$. The projective norm-graph $NG(q,t)$ was defined by Alon-R.-Szabó. The vertex set of $NG(q,t)$ is $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$. Two vertices (A,a) and (B,b) of the graph are adjacent if and only if $N(A+B) = ab$. The projective norm graphs $NG(q,t)$ are known to provide tight constructions for the Turán number of complete bipartite graphs $K_{t,s}$ with $s > (t-1)!$, in particular $NG(q,t)$ does not contain complete bipartite subgraphs isomorphic to $K_{t,(t-1)!+1}$.

In the talk we discuss some of the basic properties of projective norm graphs, including a key algebraic result from Kollár-R.-Szabó. We study $NG(q,4)$ in more detail. It does not have $K_{4,7}$ subgraphs and we prove that it does contain (many copies of) $K_{4,6}$ for any prime power $q \geq 5$. We obtain this by studying certain systems of norm equations over finite fields. A connection with Singer difference sets is established. Also, we count the copies of any fixed 3-degenerate subgraph, and find that projective norm graphs display quasirandom behaviour with respect to this parameter. Finally we describe the automorphism group of $NG(q,t)$.