Grassmann graphs of finite-rank self-adjoint operators

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The Grassmann graph Γ is the graph whose vertices are m-dimensional subspaces of a vector space, and two subspaces are adjacent (connected by an edge) if their intersection is $(m-1)$-dimensional. Classical Chow's theorem [1] describes automorphisms of Γ.

Projections in a Hilbert space can be characterized as self-adjoint idempotents in the algebra of bounded operators. Two projections (of the same rank) are adjacent if their images are adjacent subspaces. Thus, the Grassmann graph of rank-m projections and the Grassmann graph of m-dimensional subspaces in a Hilbert space, are isomorphic.

In [2] and [3] we consider the graph $\Gamma_{\mathcal{C}}$, whose vertex set is a conjugacy class $\mathcal{C}$ consisting of finite-rank self-adjoint operators on a complex Hilbert space H. Firstly, we need to extend the adjacency relation from conjugacy class of projections to $\mathcal{C}$. It is natural to require that the difference of two adjacent operators $A, B \in \mathcal{C}$ must be of minimal rank, which turns to be two. However, this condition does not guarantee that A and B have sufficiently many common eigenvectors. The requirement that the kernel and the image of $A-B$ are invariant to both A and B resolves this issue.

In the case when the operators from $\mathcal{C}$ have two eigenvalues only, we obtain the Grassmann graph formed by k-dimensional subspaces of H, where k is the smallest dimension of eigenspaces. So, this case is covered by Chow's theorem. Under the assumption that operators from $\mathcal{C}$ have more than two eigenvalues we show that every automorphism of the graph $\Gamma_{\mathcal{C}}$ is induced by a unitary or anti-unitary operator up to a permutation of eigenspaces with the same dimensions.

References

[1] W.L. Chow, *On the geometry of algebraic homogeneous spaces*, Ann. of Math. 50 (1949), 32–67.

[2] M. Pankov, K. Petelczyc, M. Żynel *Generalized Grassmann graphs associated to conjugacy classes of finite-rank self-adjoint operators*, Linear Algebra Appl. 627 (2021), 1–23.

[3] M. Pankov, K. Petelczyc, M. Żynel *Automorphisms of graphs corresponding to conjugacy classes of finite-rank self-adjoint operators*, submitted