Problems of the Miklós Schweitzer Memorial Competition, 2017.

1. Can one divide a square into finitely many triangles such that no two triangles share a side? (The triangles have pairwise disjoint interiors and their union is the square.)

2. Prove that a field K can be ordered if and only if every $A \in M_n(K)$ symmetric matrix can be diagonalized over the algebraic closure of K. (In other words, for all $n \in \mathbb{N}$ and all $A \in M_n(K)$, there exists an $S \in GL_n(\overline{K})$ for which $S^{-1}AS$ is diagonal.)

3. For every algebraic integer α define its positive degree $\deg^+(\alpha)$ to be the minimal $k \in \mathbb{N}$ for which there exists a $k \times k$ matrix with non-negative integer entries with eigenvalue α. Prove that for any $n \in \mathbb{N}$, every algebraic integer α with degree n satisfies $\deg^+(\alpha) \leq 2n$.

4. Let K be a number field which is neither $\mathbb{Q}$ nor a quadratic imaginary extension of $\mathbb{Q}$. Denote by $\mathcal{L}(K)$ the set of integers $n \geq 3$ for which we can find units $\varepsilon_1, \ldots, \varepsilon_n \in K$ for which

$$\varepsilon_1 + \cdots + \varepsilon_n = 0,$$

but $\sum_{i \in I} \varepsilon_i \neq 0$ for any nonempty proper subset I of $\{1, 2, \ldots, n\}$. Prove that $\mathcal{L}(K)$ is infinite, and that its smallest element can be bounded from above by a function of the degree and discriminant of K. Further, show that for infinitely many K, $\mathcal{L}(K)$ contains infinitely many even and infinitely many odd elements.

5. For every non-constant polynomial p, let $H_p = \{z \in \mathbb{C} \,\big|\, |p(z)| = 1\}$. Prove that if $H_p = H_q$ for some polynomials p, q, then there exists a polynomial r such that $p = r^m$ and $q = \xi \cdot r^n$ for some positive integers m, n and constant $|\xi| = 1$.

6. Let I and J be intervals. Let $\varphi, \psi : I \to \mathbb{R}$ be strictly increasing continuous functions and let $\Phi, \Psi : J \to \mathbb{R}$ be continuous functions. Suppose that $\varphi(x) + \psi(x) = x$ and $\Phi(u) + \Psi(u) = u$ holds for all $x \in I$ and $u \in J$. Show that if $f : I \to J$ is a continuous solution of the functional inequality

$$f\big(\varphi(x) + \psi(y)\big) \leq \Phi\big(f(x)\big) + \Psi\big(f(y)\big) \qquad (x, y \in I),$$

then $\Phi \circ f \circ \varphi^{-1}$ and $\Psi \circ f \circ \psi^{-1}$ are convex functions.

7. Characterize all increasing sequences (s_n) of positive reals for which there exists a set $A \subset \mathbb{R}$ with positive measure such that $\lambda(A \cap I) < \frac{s_n}{n}$ holds for every interval I with length $1/n$, where λ denotes the Lebesgue measure.

8. Let the base 2 representation of $x \in [0; 1)$ be $x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}$. (If x is dyadically rational, i.e. $x \in \left\{\frac{k}{2^n} : k, n \in \mathbb{Z}\right\}$, then we choose the finite representation.) Define function $f_n : [0; 1) \to \mathbb{Z}$ by

$$f_n(x) = \sum_{j=0}^{n-1} (-1)^{\sum_{i=0}^{j} x_i}.$$

Does there exist a function $\varphi : [0;\infty) \to [0;\infty)$ such that $\lim_{x\to\infty} \varphi(x) = \infty$ and

$$\sup_{n\in\mathbb{N}} \int_0^1 \varphi(|f_n(x)|)\mathrm{d}x < \infty\,?$$

9. Let N be a normed linear space with a dense linear subspace M. Prove that if $L_1, \dots, L_m$ are continuous linear functionals on N, then for all $x \in N$ there exists a sequence (y_n) in M converging to x satisfying $L_j(y_n) = L_j(x)$ for all $j = 1, \dots, m$ and $n \in \mathbb{N}$.

10. Let $X_1, X_2, \dots$ be independent and identically distributed random variables with distribution $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$. Let Y_1, Y_2, Y_3, and Y_4 be independent, identically distributed random variables, where $Y_1 := \sum_{k=1}^{\infty} \frac{X_k}{16^k}$. Decide whether the random variables $Y_1 + 2Y_2 + 4Y_3 + 8Y_4$ and $Y_1 + 4Y_3$ are absolutely continuous.