Problems of the Miklós Schweitzer Memorial Competition, 2010.

1. Let p be prime. Denote by $N(p)$ the number of integers x for which $1 \le x \le p$ and

$$x^x \equiv 1 \pmod{p}.$$

Prove that there exist numbers $c < 1/2$ and $p_0 > 0$ such that

$$N(p) \le p^c$$

if $p \ge p_0$.

2. Let G be a countably infinite, d -regular, connected, vertex-transitive graph. Show that there is a complete pairing in G.

3. Let $A_i, i = 1, 2, \dots, t$ be distinct subsets of the base set $\{1, 2, \dots, n\}$ complying to the following condition

$$A_i \cap A_k \subseteq A_j$$

for any $1 \le i < j < k \le t$. Find the maximum value of t.

4. Prove that if $n \ge 2$ and $I_1, I_2, \dots, I_n$ are prime ideals in a unitary commutative ring such that for any nonempty $H \subseteq \{1, 2, \dots, n\}$ the set $\sum_{h \in H} I_h$ is a prime ideal, then

$$I_2 I_3 I_4 \dots I_n + I_1 I_3 I_4 \dots I_n + \dots + I_1 I_2 \dots I_{n-1}$$

is also a prime ideal.

5. Given the vectors $v_1, \dots, v_n$ and $w_1, \dots, w_n$ in the plane with the following properties: for every $1 \le i \le n$, $|v_i - w_i| \le 1$, and for every $1 \le i < j \le n$, $|v_i - v_j| \ge 3$ and $v_i - w_i \ne v_j - w_j$. Prove that for sets $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$, the set of $V + (V \cup W)$ must have at least $cn^{3/2}$ elements ,for some universal constant $c > 0$.

6. Is there a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ for every $d \in \mathbb{R}$ we have $g_{m,d}(x) = f(x, mx+d)$ is strictly monotonic on $\mathbb{R}$ if $m \ge 0$, and not monotone on any non-empty open interval if $m < 0$?

7. Is there any sequence $(a_n)_{n=1}^{\infty}$ of non-negative numbers, for which $\sum_{n=1}^{\infty} a_n^2 < \infty$, but $\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{a_{kn}}{k} \right)^2 = \infty$?

8. Let $D \subset \mathbb{R}^2$ be a finite Lebesgue measure of a connected open set and $u : D \to \mathbb{R}$ a harmonic function. Show that it is either a constant u or for almost every $p \in D$

$$f'(t) = (\operatorname{grad} u)(f(t)), \quad f(0) = p$$

has no initial value problem(differentiable everywhere) solution to $f : [0, \infty) \to D$.

9. For each M m-dimensional closed C^∞ set , assign a $G(m)$ in some euclidean space $\mathbb{R}^q$. Denote by $\mathbb{RP}^q$ a q-dimensional real projecive space. $\mathrm{A}G(M) \subseteq \times\mathbb{RP}^q$. The set consists of (x, e) pairs for which $x \in M \subseteq \mathbb{P}^q$ and $e \subseteq \mathbb{R}^{q+1} = \mathbb{R}^q \times \mathbb{R}$ and a$(0, \dots, 0, 1) \in \mathbb{R}^{q+1}$ in a stretched $(m+1)$-dimensional linear subspace. Prove that if N is a n-dimensional closed set C^∞, then $P = G(M \times M)$ and $Q = G(M) \times G(N)$ are cobordant , that is, there exists a $(2m+2n+1)$-dimensional compact , flanged set C^∞ with a disjoint union of P and Q.

10. Consider the space $\{0,1\}^N$ with the product topology (where $\{0,1\}$ is a discrete space). Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the left-shift, ie $(Tx)(n) = x(n+1)$ for every $n \in \mathbb{N}$. Can a finite number of Borel sets be given: $B_1, \ldots, B_m \subset \{0,1\}^N$ such that

$$\left\{T^i\left(B_j\right) \mid i \in \mathbb{N}, 1 \leq j \leq m\right\}$$

the σ-algebra generated by the set system coincides with the Borel set system?

11. (Not yet translated.)