Problems of the Miklós Schweitzer Memorial Competition, 2008.

1. Let $\mathcal{H} \subset P(X)$ be a system of subsets of X and $\kappa > 0$ be a cardinal number such that every $x \in X$ is contained in less than κ members of $\mathcal{H}$. Prove, that there exists an $f : X \to \kappa$ coloring, such that every nonempty $A \in \mathcal{H}$ has a "unique" point, that is, an element $x \in A$ such that $f(x) \neq f(y)$ for all $x \neq y \in A$.

2. Let $t \geq 3$ be an integer, and for $1 \leq i < j \leq t$ let $A_{ij} = A_{ji}$ be an arbitrary subset of an n-element set X. Prove, that there exist $1 \leq i < j \leq t$ for which

$$\Big|(X \setminus A_{ij}) \cup \bigcup_{k \neq i,j} (A_{ik} \cap A_{jk})\Big| \geq \frac{t-2}{2t-2} n.$$

3. A bipartite graph on the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of vertices (that is the edges are of the form $x_i y_j$) is called tame if it has no $x_i y_j x_k y_\ell$ path ($i, j, k, \ell \in \{1, \dots, n\}$) where $j < \ell$ and $i + j > k + \ell$. Calculate the infimum of those real numbers α for which there exists a constant $c = c(\alpha) > 0$ such that for all tame graphs $e \leq c \cdot n^\alpha$, where e is the number of edges and n is half of the number of vertices.

4. Let A be a subgroup of the symmetric group S_n, and G be a normal subgroup of A. Show, that if G is transitive, then $|A : G| \leq 5^{n-1}$.

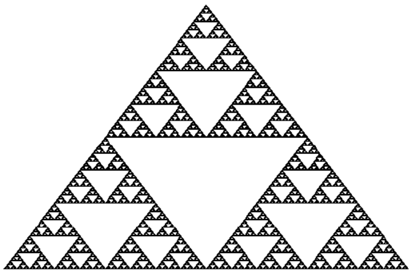
5. Let A be an infinite subset of the set of natural numbers, and denote by $\tau_A(n)$ the number of divisors of n in A. Construct a set A for which

$$\sum_{n \leq x} \tau_A(n) = x + O(\log \log x),$$

and show that there is no set for which the error term is $o(\log \log x)$ in the above formula.

6. Can you draw circles on the plane so that every line intersects at least one of them but no more than 100 of them?

7. Let $f : \mathbb{R}^1 \to \mathbb{R}^2$ be a continuous function such that $f(x) = f(x+1)$ for all x, and let $t \in [0, 1/4]$. Prove, that there exists $x \in \mathbb{R}$, such that the vector from $f(x-t)$ to $f(x+t)$ is perpendicular to the vector from $f(x)$ to $f(x+1/2)$.



8. Let S be the above Sierpinski triangle. What can we say about the Hausdorff dimension of the elevation sets $f^{-1}(y)$ for typical continuous real functions defined on S? (A property is satisfied for typical continuous real functions on S if the set of functions not having this property is of the first Baire category in the metric space of continuous $S \to \mathbb{R}$ functions with the supremum norm.)

9. For a given $\alpha > 0$ let us consider the regular, non-vanishing $f(z)$ maps on the unit disc $\{|z| < 1\}$ for which $f(0) = 1$ and $\Re z\frac{f'(z)}{f(z)} > -\alpha$ $(|z| < 1)$. Show that the range of

$$g(z) = \frac{1}{(1-z)^{2\alpha}}$$

contains the range of all other such functions. Here we consider that regular branch of $g(z)$ for which $g(0) = 1$.

10. Let V be the set of non-collinear pairs of vectors in $\mathbb{R}^3$, and H be the set of lines passing through the origin. Is it true that for every continuous map $f : V \to H$ there exists a continuous map $g : V \to \mathbb{R}^3 \setminus \{0\}$ such that $g(v) \in f(v)$ for all $v \in V$?

11. Let $\xi_1, \ldots, \xi_n$ be (not necessarily independent) random variables with normal distribution for which $E\xi_j = 0$ and $E\xi_j^2 \leq 1$ for all $1 \leq j \leq n$. Prove, that

$$E\left(\max_{1 \leq j \leq n} \xi_j\right) \leq \sqrt{2\log n}.$$

Unofficial translation by Miklós Maróti