Problems of the Miklós Schweitzer Memorial Competition, 2007.

1. Prove, that there exist subfields of $\mathbf{R}$ that are

a) non-measurable, and
b) of measure zero and continuum cardinality.

2. We partition the n-element subsets of an n^2+n+1-element set into two classes. Prove, that one of the classes contains n-many pairwise disjunct sets.

3. Denote by $\omega(n)$ the number of prime divisors of the natural number n (without multiplicities). Let

$$F(x) = \max_{n\le x} \omega(n), \qquad G(x) = \max_{n\le x} \left(\omega(n) + \omega(n^2+1)\right).$$

Prove, that $G(x) - F(x) \to \infty$ as $x \to \infty$.

4. Let p be a prime number, and $a_1, \dots, a_{p-1}$ be not necessarily distinct nonzero elements of the p-element $\mathbf{Z}_p$ (mod p) group. Prove, that each element of $\mathbf{Z}_p$ equals a sum of some of the a_i's (the empty sum is 0).

5. Let $D = \{\, (x, y) \mid x > 0,\ y \neq 0\,\}$, and let $u \in C^1(\overline{D})$ be a bounded function that is harmonic on D and for which $u = 0$ on the y-axis. Prove, that u is identically zero.

6. For which subsets $A \subset \mathbf{R}$ is it true, that whenever $0 \le x_0 < x_1 < \dots < x_n \le 1$, $n = 1, 2, \dots$, then there exist $y_j \in A$ numbers, such that $y_{j+1} - y_j > x_{j+1} - x_j$ for all $0 \le j < n$.

7. Prove, that there exist natural numbers n_k, m_k, $k = 0, 1, 2, \dots$, such that the numbers $n_k + m_k$, $k = 1, 2, \dots$ are pairwise distinct primes, and the set of linear combination of the polynomials $x^{n_k} y^{m_k}$ is dense in $C([0, 1] \times [0, 1])$ under the supremum norm.

8. For an $A = \{a_i\}_{i=0}^{\infty}$ sequence let $SA = \{a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots\}$ be the sequence of partial sums of the $a_0+a_1+a_2+\dots$ series. Does there exist a non-identically zero sequence A such that all of the sequences A, SA, SSA, $SSSA$, $\dots$ are convergent?

9. Let A and B be two triangles on the plane, such that the interior of both contains the origin, and for each circle C_r centered at the origin $|C_r \cap A| = |C_r \cap B|$ (where $|\cdot|$ is the arc-length measure). Prove, that A and B are congruent. Does this statement remain true if the origin is on the border of A or B?

10. Let $\zeta_1, \zeta_2, \dots$ be identically distributed, independent real-valued random variables with expected value 0. Suppose that the $\Lambda(\lambda) := \log \mathbb{E} \exp(\lambda\zeta_i)$ logarithmic moment-generating function always exists for $\lambda \in \mathbb{R}$ ($\mathbb{E}$ is the expected value). Furthermore, let $G\colon \mathbb{R} \to \mathbb{R}$ be a function such that $G(x) \le \min(|x|, x^2)$. Prove that for small $\gamma > 0$ the following sequence is bounded:

$$\left\{ \mathbb{E} \exp \left(\gamma l G \left(\frac{1}{l} (\zeta_1 + \ldots + \zeta_l) \right) \right) \right\}_{l=1}^{\infty}.$$

Unofficial translation by Miklós Maróti