Miklós Schweitzer Memorial Competition in Mathematics
29 October — 8 November 2004

1. The Lindelöf number $L(X)$ of a topological space X is the least infinite cardinal λ with the property that every open covering of X has a subcovering of cardinality at most λ. Prove that if every non-countably infinite subset of a first countable space X has a point of condensation, then $L(X) = \sup L(A)$, where A runs over the separable closed subspaces of X.
   (A point of condensation of a subset $H \subseteq X$ is a point $x \in X$ such that any neighbourhood of x intersects H in a non-countably infinite set.)

2. Write $t(G)$ for the number of complete quadrilaterals in the graph G and $e_G(S)$ for the number of edges spanned by a subset S of vertices of G. Let G_1, G_2 be two (simple) graphs on a common underlying set V of vertices, $|V| = n$, and assume that $|e_{G_1}(S) - e_{G_2}(S)| \leq n^2/1000$ holds for any subset $S \subseteq V$. Prove that $|t(G_1) - t(G_2)| \leq n^4/1000$.

3. Prove that there is a constant $c > 0$ such that for any $n \geq 3$ there exists a planar graph G with n vertices such that every straight-edged plane embedding of G has a pair of edges with ratio of lengths at least cn.

4. Determine all totally multiplicative and non-negative functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that if $a, b \in \mathbb{Z}$ and $b \neq 0$, then there exist integers q and r such that $a = qb + r$ and $f(r) < f(b)$.

5. Let G be a non-solvable finite group and let $\varepsilon > 0$. Show that there exist a positive integer k and a word $w \in F_k$ such that w assumes the value 1 with probability less than ε when its k arguments are considered to be independent and uniformly distributed random variables with values in G. (We write F_k for the free group generated by k elements.)

6. Is it true that if the perfect set $F \subset [0, 1]$ is of zero Lebesgue measure then those functions in $C^1[0, 1]$ which are one-to-one on F form a dense subset of $C^1[0, 1]$?
   (We use the metric
   $$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |f'(x) - g'(x)|$$
   to define the topology in the space $C^1[0, 1]$ of continuously differentiable real functions on $[0, 1]$.)

7. Suppose that the closed subset K of the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is symmetric with respect to the origin and separates any two antipodal points in $S^2 \backslash K$. Prove that for any positive ε there exists a homogeneous polynomial P of odd degree such that the Hausdorff distance between

$$Z(P) = \{(x, y, z) \in S^2 : P(x, y, z) = 0\}$$

and K is less than ε.

8. Prove that for any $0 < \delta < 2\pi$ there exists a number $m \geq 1$ such that for any positive integer n and unimodular complex numbers $z_1, \dots, z_n$ with $z_1^\nu + \dots + z_n^\nu = 0$ for all integer exponents $1 \leq \nu \leq m$, any arc of length δ of the unit circle contains at least one of the numbers $z_1, \dots, z_n$.

9. Let F be a smooth (i.e., C^∞) closed surface. Call a continuous map $f : F \to \mathbb{R}^2$ an *almost-immersion* if there exists a smooth closed embedded curve γ (possibly disconnected) in F such that f is smooth and of maximal rank (i.e., rank 2) on $F \backslash \gamma$, and each point $p \in \gamma$ admits local coordinate charts (x, y) and (u, v) about p and $f(p)$, respectively, such that the coordinates of p and $f(p)$ are zero and the map f is given by $(x, y) \mapsto (u, v)$, $u = |x|$, $v = y$.
Determine the genera of those smooth, closed, connected, orientable surfaces F that admit an almost-immersion in the plane with the curve γ having a given positive number n of connected components.

10. Let $\mathcal{N}_p$ stand for a p dimensional random variable of standard normal distibution. For $a \in \mathbb{R}^p$, let $H_p(a)$ stand for the expectation $E|\mathcal{N}_p + a|$. For $p > 1$, prove that

$$H_p(a) = (p-1) \int_0^\infty H_1\left(\frac{|a|}{\sqrt{1+r^2}}\right) \frac{r^{p-2}}{(1+r^2)^{\frac{p}{2}}} dr.$$

The deadline for submitting solutions is 12:00 (CET) 8 November, 2004. If the participant relies on knowledge not included in the standard curriculum, then (s)he should cite the exact source. For further information see the homepage `http://www.cs.elte.hu/~schw04`.