Problems of the Miklós Schweitzer Memorial Competition
November 8–18, 2002

1. For an arbitrary ordinal number α let $H(\alpha)$ denote the set of functions $f : \alpha \to \{-1, 0, 1\}$ that map all but finitely many elements of α to 0. Order $H(\alpha)$ according to the last difference, that is, for $f, g \in H(\alpha)$ let $f \prec g$ if $f(\beta) < g(\beta)$ holds for the maximum ordinal number $\beta < \alpha$ with $f(\beta) \neq g(\beta)$. Prove that the ordered set $(H(\alpha), \prec)$ is scattered (i.e. it does not contain a subset isomorphic to the set of rational numbers with the usual order), and that any scattered order type can be embedded into some $(H(\alpha), \prec)$.

2. Let G be a simple k-edge-connected graph on n vertices and let u and v be different vertices of G. Prove that there exist k edge-disjoint paths from u to v each having at most $\dfrac{20n}{k}$ edges.

3. Put $\mathbf{A} = \{\text{yes}, \ \text{no}\}$. A function $f : \mathbf{A}^n \to \mathbf{A}$ is called a *decision function* if

   (a) the value of the function changes if we change all of its arguments; and

   (b) the value does not change if we replace any of the arguments by the function value.

   A function $d : \mathbf{A}^n \to \mathbf{A}$ is called a *dictatoric function*, if there is an index i such that the value of the function equals its ith argument.

   The *democratic function* is the function $m : \mathbf{A}^3 \to \mathbf{A}$ that outputs the majority of its arguments.

   Prove that any decision function is a composition of dictatoric and democratic functions.

4. For a given natural number n, consider those sets $A \subseteq \mathbb{Z}_n$ for which the equation $xy = uv$ has no other solution in the residual classes $x, y, u, v \in A$ than the trivial solutions $x = u, \ y = v$ and $x = v, \ y = u$. Let $g(n)$ be the maximum of the size of such sets A. Prove that

$$\limsup_{n\to\infty} \frac{g(n)}{\sqrt{n}} = 1 \ .$$

5. Denote by $\lambda(H)$ the Lebesgue outer measure of $H \subseteq [0, 1]$. The horizontal and vertical sections of the set $A \subseteq [0, 1] \times [0, 1]$ are denoted by A^y and A_x respectively; that is, $A^y = \{x \in [0, 1] : (x, y) \in A\}$ and $A_x = \{y \in [0, 1] : (x, y) \in A\}$ for all $x, \ y \in [0, 1]$.

   (a) Is there a decomposition $A \cup B$ of the unit square $[0, 1] \times [0, 1]$ such that A^y is the union of finitely many segments of total length less than $1/2$ and $\lambda(B_x) \leq 1/2$ for all $x, \ y \in [0, 1]$?

   (b) Is there a decomposition $A \cup B$ of the unit square $[0, 1] \times [0, 1]$ such that A^y is the union of finitely many segments of total length not greater than $1/2$ and $\lambda(B_x) < 1/2$ for all $x, \ y \in [0, 1]$?

6. Let $K \subseteq \mathbb{R}$ be compact. Prove that the following two statements are equivalent to each other.

   (a) For each point x of K we can assign an uncountable set $F_x \subseteq \mathbb{R}$ such that

   $$\operatorname{dist}(F_x, F_y) \geq |x - y|$$

   holds for all $x, y \in K$;

   (b) K is of measure zero.

7. Let the complex function $F(z)$ be regular on the punctuated disk $\{0 < |z| < R\}$. By a *level curve* we mean a component of the level set of $\operatorname{Re} F(z)$, that is, a maximal connected set on which $\operatorname{Re} F(z)$ is constant. Denote by $A(r)$ the union of those level curves that are entirely contained in the punctuated disk $\{0 < |z| < r\}$. Prove that if the number of components of $A(r)$ has an upper bound independent of r then $F(z)$ can only have a pole type singularity at 0.

8. Prove that there exists an absolute constant c such that any set H of n points of the plane in general position can be coloured with $c \cdot \log n$ colours in such a way that any disk of the plane containing at least one point of H intersects some colour class of H in exactly one point.

9. Let M be a connected, compact C^∞-differentiable manifold, and dentote the vector space of smooth real functions on M by $C^\infty(M)$. Let the subspace $V \leq C^\infty(M)$ be invariant under C^∞-diffeomorphisms of M, that is, let $f \circ h \in V$ for every $f \in V$ and for every C^∞-diffeomorphism $h : M \to M$. Prove that if V is different from the subspaces $\{0\}$ and $C^\infty(M)$ then V only contains the constant functions.

10. Let $X_1, X_2, \ldots$ be independent random variables of the same distribution such that their joint distribution is discrete and is concentrated on infinitely many different values. Let a_n denote the probability that $X_1, \ldots, X_{n+1}$ are all different on the condition that $X_1, \ldots, X_n$ are all different ($n \geq 1$). Show that

    (a) a_n is strictly decreasing and tends to 0 as $n \to \infty$; and

    (b) for any sequence $1 \leq f(1) < f(2) < \ldots$ of positive integers the joint distribution of $X_1, X_2, \ldots$ can be chosen such that

    $$\limsup_{n \to \infty} \frac{a_{f(n)}}{a_n} = 1$$

    holds.

The deadline for submitting solutions to the problems is November the 18th, 2002./ 12h (CET). If the participant uses some knowledge that is not contained in the standard curriculum, then (s)he should cite the exact source. For further information see the homepage `http://www.cs.elte.hu/~schw02` .