LARGE-AMPLITUDE PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH DELAYED MONOTONE POSITIVE FEEDBACK

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Abstract: The aim of this paper is to show that the structure of the global attractor for delayed monotone positive feedback can be more complicated than the union of spindle-like structures between consecutive stable equilibria with respect to the pointwise ordering. Large amplitude periodic orbits – in the sense that they are not between two consecutive stable equilibria – are constructed for nonlinearities close to a step function. For some nonlinearities there are exactly two large amplitude periodic orbits. By describing the unstable sets of these periodic orbits, a complete picture is obtained about the global attractor outside the spindle-like structures.

Suggested running head: Large-Amplitude Periodic Solutions for Delayed Feedback

Key words: delay differential equation, positive feedback, periodic orbit, large amplitude, discrete Lyapunov functional, hyperbolicity, return map, heteroclinic orbit

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1. Introduction

The delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

with $\mu \geq 0$ and smooth monotone nonlinearity $f : \mathbb{R} \to \mathbb{R}$ appears in several applications, see e.g. [6, 7, 11, 17, 31] and the references therein.

The natural phase space for Eq. (1.1) is $C = C([-1, 0], \mathbb{R})$ equipped with the supremum norm. For any $\varphi \in C$, there is a unique solution $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ of (1.1). For

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each $t \geq 0$, $x_t^\varphi \in C$ is defined by $x_t^\varphi(s) = x^\varphi(t+s)$, $-1 \leq s \leq 0$. The map

$$\Phi : [-1, \infty) \times C \ni (t, \varphi) \mapsto x_t^\varphi \in C$$

is a continuous semiflow. Very much is known about the global dynamics of Eq. (1.1). A discrete Lyapunov functional, as a key technical tool, combined with several other dynamical system methods makes is possible to prove a Poincaré–Bendixson type result for (1.1) [22] and to obtain a lot of information about the structure of the global attractor [13, 14, 15, 16, 17, 20, 23, 24, 29]. For some particular nonlinearities like $f(x) = \alpha \tanh(\beta x)$ or $f(x) = \alpha \tan^{-1}(\beta x)$ with $\alpha \neq 0$ and $\beta > 0$, a complete picture is available [17]. However, for most of the nonlinearities such a nice description is not known. A famous example is Wright's equation: $\mu = 0$, $f(x) = -\alpha(e^x - 1)$, $\alpha > 0$.

Assume (see Fig. 1)

(H1) $\mu > 0$, $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

are five consecutive zeros of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$ with $f'(\xi_j) < \mu < f'(\xi_k)$ for $j \in \{-2, 0, 2\}$ and $k \in \{-1, 1\}$.

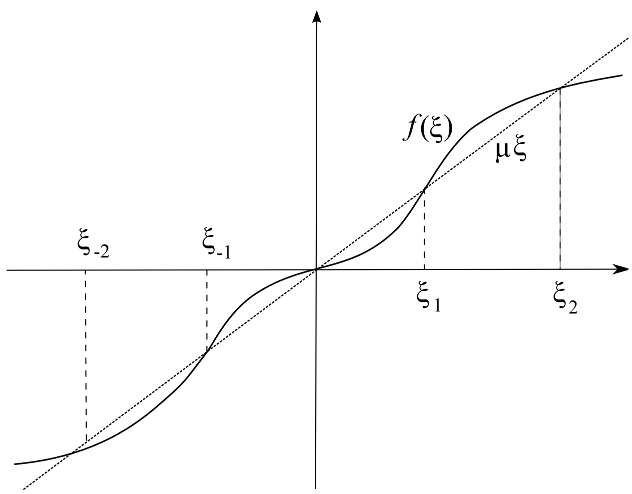

Figure 1. A feedback function satisfying condition (H1)

Under hypothesis (H1), $\hat{\xi}_j \in C$ defined by $\hat{\xi}_j(s) = \xi_j$, $-1 \leq s \leq 0$, is an equilibrium point of Φ for $j \in \{-2, -1, 0, 1, 2\}$. In addition, $\hat{\xi}_{-2}, \hat{\xi}_0, \hat{\xi}_2$ are stable and $\hat{\xi}_{-1}, \hat{\xi}_1$ are unstable. By the monotone property of f, the subsets

$$C_{j,k} = \{\varphi \in C : \ \xi_j \leq \varphi(s) \leq \xi_k, \ -1 \leq s \leq 0\}$$

of the phase space C with $j \in \{-2, 0\}$ and $k \in \{0, 2\}$ are positively invariant under the semiflow Φ. The structures of the global attractors $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ of the restrictions $\Phi|_{[0,\infty)\times C_{-2,0}}$ and $\Phi|_{[0,\infty)\times C_{0,2}}$, respectively, are (at least partially) well understood, see e.g. [13, 14, 15, 16, 17, 18]. In particular cases, $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ have spindle-like structures described in [13, 16, 17, 18]: $\mathcal{A}_{0,2}$ is the closure of the unstable set of $\hat{\xi}_1$ containing the equilibrium points $\hat{\xi}_0$, $\hat{\xi}_1$, $\hat{\xi}_2$, periodic orbits in $C_{0,2}$ and heteroclinic orbits among them; and analogously for $\mathcal{A}_{-2,0}$.

Let $\mathcal{A}$ denote the global attractor of the restriction $\Phi|_{[0,\infty)\times C_{-2,2}}$. It is easy to see that if (H1) holds and $\xi_{-2}, \xi_{-1}, 0, \xi_1, \xi_2$ are the only zeros of $-\mu\xi + f(\xi)$, then $\mathcal{A}$ is the global attractor of Φ. The problem, whether under hypothesis (H1) the equality

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \tag{1.2}$$

holds or not, arose in [17], see Fig. 2.

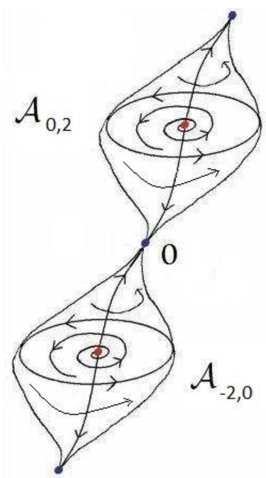

FIGURE 2. $\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$

The main result of this paper is that $\mathcal{A}$ can be more complicated than given by (1.2). We construct examples so that Eq. (1.1) with assumption (H1) has periodic orbits in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$. The periodic solutions defining these periodic orbits oscillate slowly around 0 and have large amplitudes in the following sense.

A periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) is called a large amplitude periodic solution if $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. A solution $x : \mathbb{R} \to \mathbb{R}$ is slowly oscillatory if for each t, the restriction $x|_{[t-1,t]}$ has 1 or 2 sign changes. A solution $x : \mathbb{R} \to \mathbb{R}$ is called slowly oscillatory around ξ_j if $\mathbb{R} \ni t \mapsto x(t) - \xi_j \in \mathbb{R}$ is slowly oscillatory. Note that here slow oscillation is different from the usual one used for equations with negative feedback condition [6, 29]. A large-amplitude slowly oscillatory periodic solution $x : \mathbb{R} \to \mathbb{R}$ will be abbreviated as an LSOP solution. We say that an LSOP solution $x : \mathbb{R} \to \mathbb{R}$ is normalized if $x(-1) = 0$, and for some $\eta > 0$, $x(s) > 0$ for all $s \in (-1, -1 + \eta)$.

Theorem 1.1. *There exist μ and f satisfying (H1) such that Eq. (1.1) has exactly two normalized LSOP solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$. For the ranges of p and q, $p(\mathbb{R}) \subsetneq q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$ holds. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

are hyperbolic and unstable with 2 and 1 Floquet multipliers outside the unit circle, respectively.

In the situation of Theorem 1.1, let $\mathcal{W}^u(\mathcal{O}_p)$ and $\mathcal{W}^u(\mathcal{O}_q)$ denote the unstable sets of $\mathcal{O}_p$ and $\mathcal{O}_q$, respectively [6, 17].

The nonlinearity f and constant μ in Theorem 1.1 are given so that there exist periodic solutions oscillating slowly around ξ_1 and ξ_{-1} with ranges in $(0, \xi_2)$ and $(\xi_{-2}, 0)$, respectively [17]. Among these periodic solutions there are x^1 and x^{-1} so that the ranges $x^1(\mathbb{R})$ and $x^{-1}(\mathbb{R})$ are maximal in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly around ξ_1 with range in $(0, \xi_2)$, analogously for x^{-1}, see Proposition 2.7. Set

$$\mathcal{O}_1 = \left\{x_t^1 : t \in \mathbb{R}\right\} \text{ and } \mathcal{O}_{-1} = \left\{x_t^{-1} : t \in \mathbb{R}\right\}.$$

Under further restriction on f, the dynamics in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ can be completely described.

Theorem 1.2. *One may set μ and f satisfying (H1) such that the statement of Theorem 1.1 holds, and for the global attractor $\mathcal{A}$ we have the equality*

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q).$$

Moreover, the dynamics on $\mathcal{W}^u(\mathcal{O}_p)$ and $\mathcal{W}^u(\mathcal{O}_q)$ is as follows.
For each $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$, the omega limit set $\omega(\varphi)$ is either $\left\{\hat{\xi}_{-2}\right\}$ or $\left\{\hat{\xi}_2\right\}$, and there exist heteroclinic connections from $\mathcal{O}_q$ to $\left\{\hat{\xi}_{-2}\right\}$ and to $\left\{\hat{\xi}_2\right\}$.
For each $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, $\omega(\varphi)$ is one of the sets $\left\{\hat{\xi}_{-2}\right\}$, $\{\hat{0}\}$, $\left\{\hat{\xi}_2\right\}$, $\mathcal{O}_q$, $\mathcal{O}_1$, $\mathcal{O}_{-1}$. There are heteroclinic connections from $\mathcal{O}_p$ to $\left\{\hat{\xi}_{-2}\right\}$, $\{\hat{0}\}$, $\left\{\hat{\xi}_2\right\}$, $\mathcal{O}_q$, $\mathcal{O}_1$ and $\mathcal{O}_{-1}$.

The system of connecting orbits is represented in Fig. 3. The dashed arrows represent heteroclinic connections in $\mathcal{A}_{-2,0}$ and in $\mathcal{A}_{0,2}$, while the solid ones represent connecting orbits given by Theorem 1.2.

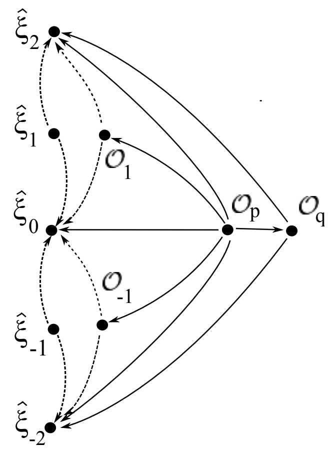

FIGURE 3. Connecting orbits

In Theorems 1.1-1.2 the nonlinear map f is close to the step function $f^{K,0}$ parametrized by $K > 0$ and given by $f^{K,0}(x) = 0$ for $|x| \leq 1$, and $f^{K,0}(x) = K\text{sgn}(x)$ for $|x| > 1$. Equations with such nonlinearity model neural networks of identical neurons that do not react upon small feedback; the feedback has to reach a certain threshold value to have a considerable effect [8]. Our result may have interesting consequences for the dynamics of neural networks with the above property. See [2, 3, 4, 5, 31] for a bistable situation.

Suppose f is odd and satisfies (H1). It follows from results in [22] that if $x : \mathbb{R} \to \mathbb{R}$ is an LSOP solution of Eq. (1.1) with minimal period $\omega > 0$, then the following statements hold.

(i) $\omega \in (1, 2)$.

(ii) Solution x is of special symmetry meaning that relation $x(t + \omega/2) = -x(t)$ holds for all $t \in \mathbb{R}$.

(iii) Solution x is of monotone type in the following sense: if $t_0 < t_1 < t_0 + \omega$ is set so that $x(t_0) = \min_{t\in\mathbb{R}} x(t)$ and $x(t_1) = \max_{t\in\mathbb{R}} x(t)$, then x is nondecreasing on $[t_0, t_1]$ and nonincreasing on $[t_1, t_0 + \omega]$.

This motivates the next definition. We say a periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with feedback function $f^{K,0}$, $K > 0$, is an LSOP solution if properties (i), (ii) and (iii) hold for x.

For Eq. (1.1) with $\mu = 1$ and $f = f^{K,0}$, the LSOP solutions are described in Theorem 6.5: there is no such solution if $K < K^* \approx 6.8653$ and there are two for $K > K^*$ (up to time translation). It can be also verified that there is exactly one LSOP solution for $K = K^*$. This is the starting point of our construction. The implicit function theorem and perturbations of Poincaré maps from [19] can be applied to find exactly two LSOP orbits of Eq. (1.1) for $\mu = 1$ and nonlinearities that satisfy (H1) and are close to $f^{K,0}$ with $K > K^*$. We prove only the case $K = 7$, which suffices for the proof of Theorem 1.1. Our results and numerical examples suggest that the LSOP orbits appear in a saddle-node-like bifurcation. However, it remains an open problem to understand this phenomenon.

The paper is organized as follows. The preliminary Section 2 lists definitions and notations. The notion of LSOP solutions is extended for a slightly wider range of feedback functions including smooth approximations of $f^{K,0}$. A discrete Lyapunov functional of Mallet-Paret and Sell [21] is introduced, their feedback inequality is weakened in order to allow not strictly monotone nonlinearities as well. It is also shown that for certain nonlinearities satisfying (H1), there are periodic solutions with maximal ranges in $(\xi_{-2}, 0)$ and $(0, \xi_2)$ oscillating slowly around ξ_{-1} and ξ_1, respectively.

Section 3 introduces a smooth approximation $f^{K,\varepsilon}$, $\varepsilon \in [0,1)$, of the step function $f^{K,0}$. Fix $K > 3$. We define an open set U^1 in $(0,1)^3 \times [0,1)$ and a continuous map $\Sigma : U^1 \to C$ so that for $\varepsilon > 0$ small, $U^1_\varepsilon \ni a \mapsto \Sigma(a,\varepsilon) \in C$ is smooth and its derivative is injective (see Proposition 3.7), where U^1_ε denotes the set $\left\{a \in (0,1)^3 : (a,\varepsilon) \in U^1\right\}$. Consequently, $\Sigma\left(U^1_\varepsilon \times \{\varepsilon\}\right)$ is a 3-dimensional C^1-submanifold of C. There exists an open subset U^3 of U^1 such that if $\mu = 1$ and $f = f^{K,\varepsilon}$, then for all $(a,\varepsilon) \in U^3$, the solution $x^{\Sigma(a,\varepsilon)} : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1) returns into $\Sigma\left(U^1_\varepsilon \times \{\varepsilon\}\right)$, i. e., there exists a minimal $t > 0$ with $x_t^{\Sigma(a,\varepsilon)} \in \Sigma\left(U^1_\varepsilon \times \{\varepsilon\}\right)$. This induces a smooth map $F : U^3 \to \mathbb{R}^3$ so that for all $(a,\varepsilon) \in U^3$, we have $F(a,\varepsilon) = b$ if $x_t^{\Sigma(a,\varepsilon)} = \Sigma(b,\varepsilon)$ with a minimal $t > 0$. If $F(a,\varepsilon) = a$ holds for some $(a,\varepsilon) \in U^3$, then the solution $x^{\Sigma(a,\varepsilon)}$ of Eq. (1.1) with $\mu = 1$ and $f = f^{K,\varepsilon}$ is an LSOP solution. Therefore the problem of finding LSOP solutions is reduced to a 3-dimensional fixed point equation depending on parameter ε. Proposition 3.8 shows that there is $K^* \approx 6.8653$ so that for $K > K^*$, equation $F(a,0) = a$ has a unique solution a^* in $U^3_0 = \left\{a \in (0,1)^3 : (a,0) \in U^3\right\}$. The fixed point a^* is hyperbolic; it is rigorously checked for $K = 7$. Then the implicit function theorem gives that if $K = 7$, then equation $F(a,\varepsilon) = a$ has a solution $a^*(\varepsilon)$ in $U^3_\varepsilon = \left\{a \in (0,1)^3 : (a,\varepsilon) \in U^3\right\}$ for small $\varepsilon > 0$ so that $D_a F(a^*(\varepsilon), \varepsilon)$ is hyperbolic. Analogously to the above construction, Subsection 3.2 gives another LSOP solution of (1.1) with $\mu = 1$ and $f = f^{7,\varepsilon}$ for $\varepsilon > 0$ small.

Other examples, in which the problem of finding periodic solutions is reduced to a finite dimensional fixed point problem, are found e.g in [19, 27, 28]. However, the corresponding return maps in [27, 28] are contractions, and the obtained periodic orbits are stable. This is not the case here, thus we cannot apply any contraction mapping theorem.

Section 4 shows that the hyperbolicity of the fixed points of the 3-dimensional maps of Section 3 guarantees the hyperbolicity of the corresponding LSOP solutions of Eq. (1.1) with $\mu = 1$ and $f = f^{7,\varepsilon}$, $\varepsilon > 0$ small, see Proposition 4.3. The key fact toward the proof is that a small neighborhood of the fixed point $\Sigma(a^*(\varepsilon), \varepsilon)$ in a hyperplane of C is mapped into the 3-dimensional submanifold $\Sigma(U_\varepsilon^3 \times \{\varepsilon\})$ by a suitable Poincaré return map (Proposition 4.1). The hyperbolicity of these LSOP solutions together with a result in [19] guarantee the existence of LSOP solutions for all nonlinearities f satisfying (H1) that are close to $f^{7,\varepsilon}$, $\varepsilon > 0$ small, in C^1-norm. Thereby the existence of the two LSOP solutions in Theorem 1.1 is verified.

The conception that the hyperbolicity of certain periodic orbits can be verified via showing the hyperbolicity of fixed points of suitable finite dimensional maps also appears in paper [12] of Kennedy. This paper considers state-dependent delay equations with feedback functions that are close to $f(x) = -\text{sgn}(x)$ outside a small neighborhood of 0.

Section 5 contains preparatory results toward the exact number of LSOP solutions. Propositions 5.1 and 5.2 prove monotone and symmetry properties of periodic solutions of (1.1). The C^1-smoothness and strict monotonicity from [22] is weakened slightly. The technical result of Proposition 5.3 shows that all LSOP solutions of (1.1) with $\mu = 1$ and $f = f^{7,\varepsilon}$, $\varepsilon > 0$ small, have nice regulatory properties.

Section 6 studies the exact number of LSOP solutions for the step function nonlinearity $f^{K,0}$, $K > 0$, then for $f^{7,\varepsilon}$, $\varepsilon > 0$ small, and finally for functions f close to $f^{7,\varepsilon}$. Summarizing the above results, Theorem 1.1 is obtained.

The next section excludes the existence of periodic solutions oscillating rapidly around 0.

Section 8 completes the proof of Theorem 1.2. The existence of heteroclinic orbits from $\mathcal{O}_p$, where p is the LSOP solution with smaller range, is based on the fact that the local unstable manifold $\mathcal{W}^u(p_0)$ of a Poincaré return map at its fixed point p_0 is 2-dimensional, and it is separated into two parts by its 1-dimensional leading unstable manifold $\mathcal{W}_1^u(p_0)$. Discrete Lyapunov functionals around $\xi_{-1}, 0, \xi_1$, information on eigenfunctions of the derivative of the Poincaré map associated with the two eigenvalues outside the unit circle, monotone property of the semiflow, and elementary topological arguments yield the result.

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2. Preliminaries

Notation. The natural phase space for Eq. (1.1) is the space $C = C([-1,0],\mathbb{R})$ of continuous real functions defined on $[-1,0]$ equipped with the supremum norm $\|\varphi\| = \sup_{-1\leq s\leq 0}|\varphi(s)|$. $C^1 = C^1([-1,0],\mathbb{R})$ is the space of continuously differentiable functions from $[-1,0]$ into $\mathbb{R}$ with norm $\|\varphi\|_{C^1} = \|\varphi\| + \|\varphi'\|$.

For $D \subseteq \mathbb{R}$ open, $C_b^1(D,\mathbb{R})$ denotes the space of bounded continuously differentiable functions $g : D \to \mathbb{R}$ with bounded first derivative together with norm $\|g\|_{C_b^1} = \sup_{x\in D}|g(x)| + \sup_{x\in D}|g'(x)|$.

For Banach spaces E and F over $\mathbb{R}$, the space of bounded linear operators is denoted by $\mathcal{L}(E,F)$.

For a simple closed curve $c : [a,b] \to \mathbb{R}^2$, $\operatorname{int}(c)$ and $\operatorname{ext}(c)$ denote the interior and exterior, i. e., the bounded and unbounded component of $\mathbb{R}^2 \setminus c([a,b])$, respectively.

If $U \subset \mathbb{R}^m$, $m \geq 1$, then $\operatorname{bd}U$ is for the boundary of U.

For an interval $I \subset \mathbb{R}$, we define

$$I + [-1,0] = \{t \in \mathbb{R} : t = t_1 + t_2 \text{ with } t_1 \in I,\ t_2 \in [-1,0]\}.$$

If $I \subset \mathbb{R}$ is an interval, $u : I \to \mathbb{R}$ is continuous, then for $[t-1,t] \subset I$, $u_t \in C$ is given by $u_t(s) = u(t+s)$, $-1 \leq s \leq 0$.

Definition of solution. In the sequel we consider Eq. (1.1) with smooth and nonsmooth (e.g. step function) nonlinearities and linear variational equations as well. This requires a slightly more general form of equation and a more general definition of solutions.

Consider the equation

$$\dot{y}(t) = g(t, y_t) \tag{2.1}$$

assuming that $g : \mathbb{R} \times C \to \mathbb{R}$ satisfies the condition: for each interval $I \subset \mathbb{R}$ and each continuous function $u : I + [-1, 0] \to \mathbb{R}$, the map $I \ni t \mapsto g(t, u_t) \in \mathbb{R}$ is locally integrable (i. e., integrable on compact subintervals of I). Then for given $t_0 \in \mathbb{R}$ and $0 < a \leq \infty$, a function $y : [t_0 - 1, t_0 + a) \to \mathbb{R}$ is called a solution of (2.1) on $[t_0 - 1, t_0 + a)$ if y is continuous and

$$y(t) = y(t_0) + \int_{t_0}^{t} g(s, y_s)\, \mathrm{d}s$$

holds for all $t \in [t_0, t_0 + a)$. A function $y : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (2.1) on $\mathbb{R}$ if it is a solution of (2.1) on $[t_0 - 1, \infty)$ for all $t_0 \in \mathbb{R}$.

If $y : [t_0 - 1, t_0 + a) \to \mathbb{R}$ is a solution of (2.1) on $[t_0 - 1, t_0 + a)$ and for some $(\alpha, \beta) \subset (t_0, t_0 + a)$, the map $(\alpha, \beta) \ni t \mapsto g(t, y_t) \in \mathbb{R}$ is continuous, then it is clear that y is continuously differentiable on (α, β), moreover, (2.1) holds for all $t \in (\alpha, \beta)$.

If $y : [t_0 - 1, t_0 + a) \to \mathbb{R}$ is a solution of (2.1), then obviously y is absolutely continuous on $[t_0, t_0 + a)$, and (2.1) holds almost everywhere on $[t_0, t_0 + a)$.

In the particular case

$$g(t, \varphi) = -\mu\varphi(0) + h(t, \varphi(-1)), \quad (t, \varphi) \in \mathbb{R} \times C,$$

for some $\mu \in \mathbb{R}$ and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that g satisfies the above local integrability condition, for each $\varphi \in C$ a unique solution $y : [-1, \infty) \to \mathbb{R}$ with $y_0 = \varphi$ can be given by the method of steps. Set $y(t) = \varphi(t)$ for $-1 \leq t \leq 0$. Suppose that a continuous $y : [-1, n]$ is already given for some $n \geq 0$. Then for $t \in [n, n+1]$, define

$$y(t) = e^{-\mu(t-n)} y(n) + \int_{n}^{t} e^{-\mu(t-s)} h(s, y(s-1))\, \mathrm{d}s.$$

Then $y|_{[n,n+1]}$ is absolutely continuous and (2.1) holds almost everywhere on $[n, n+1]$. It is easy to see that this construction gives the unique solution $y^{\varphi} : [-1, \infty) \to \mathbb{R}$ with $y_0^{\varphi} = \varphi$.

Semiflow. Now assume $g(t, \varphi) = -\mu\varphi(0) + f(\varphi(-1))$ with $\mu \in \mathbb{R}$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. Then the solutions of Eq. (1.1) define the continuous semiflow

$$\Phi : \mathbb{R}^+ \times C \ni (t, \varphi) \mapsto x_t^{\varphi} \in C.$$

All maps $\Phi(t, \cdot) : C \to C$, $t \geq 1$, are compact and all maps $C \ni \varphi \mapsto \Phi(t, \varphi) \in C^1$, $t \geq 1$, are continuous.

A set $M \subset C$ is called positively invariant under Φ if $\Phi(t, M) \subseteq M$ for all $t \geq 0$.

Limit sets. If $\varphi \in C$ and $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ is a bounded solution of Eq. (1.1), then the ω-limit set

$$\omega(\varphi) = \{\psi \in C : \text{ there exists a sequence } (t_n)_0^{\infty} \text{ in } [0, \infty) \\ \text{with } t_n \to \infty \text{ and } \Phi(t_n, \varphi) \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected and invariant in the sense that for every $\psi \in \omega(\varphi)$, there is a solution $x : \mathbb{R} \to \mathbb{R}$ with $x_0 = \psi$ and $x_t \in \omega(\varphi)$ for all $t \in \mathbb{R}$. For a solution $x : \mathbb{R} \to \mathbb{R}$ such that $x|_{(-\infty,0]}$ is bounded, the α-limit set

$$\alpha(x) = \{\psi \in C : \text{ there exists a sequence } (t_n)_0^{\infty} \text{ in } \mathbb{R} \\ \text{with } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected and invariant.

Poincaré return maps. Assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ in Eq. (1.1). Let $p : \mathbb{R} \to \mathbb{R}$ be a periodic solution of Eq. (1.1), and $\omega > 1$ be the minimal period of p. Let a closed linear subspace $H \subset C$ of codimension 1 be given so that $p_0 \in H$ and $\dot{p}_0 \notin H$. An application of The implicit function theorem yields a convex bounded open neighborhood N of 0 in H, $\nu \in (0, \omega)$ and a C^1-map $\gamma : \{p_0\} + N \to (\omega - \nu, \omega + \nu)$ with $\gamma(p_0) = \omega$ so that for each $(t, \varphi) \in (\omega - \nu, \omega + \nu) \times (\{p_0\} + N)$, segment x_t^{φ} belongs to H if and only if $t = \gamma(\varphi)$ ([6], Appendix I in [17], [19]). The Poincaré map is set

$$P : \{p_0\} + N \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in H.$$

Then P is continuously differentiable and has fixed point p_0. In addition, P depends smoothly on the right hand side of Eq. (1.1) [19].

Map $DP(p_0) : H \to H$ is a compact operator. The spectrum σ of $DP(p_0)$ is countable with one possible accumulation point at 0. All the nonzero points in σ are eigenvalues of finite multiplicity. Periodic solution p is said to be hyperbolic if p_0 is a hyperbolic fixed point of P, that is $DP(p_0)$ has no eigenvalues on the unit circle in $\mathbb{C}$. This hyperbolicity is the same as the one defined by the spectrum of the monodromy operator ([6, 17]). The nonzero points of σ and 1 are called Floquet multipliers.

The following proposition is a particular case of a more general result of Lani-Wayda [19].

Proposition 2.1. *Assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ and p is a hyperbolic periodic solution of Eq. (1.1) with minimal period $\omega > 1$. Let $D \subset \mathbb{R}$ be open with $\{p(t) : t \in [0, \omega)\} \subset D$. Then there exist an open ball $B \subset C_b^1(D, \mathbb{R})$ centered at f, an open neighborhood $V \subset N$*

of 0 *in* H *and a* C^1*-function* $\chi : B \to \{p_0\} + V \subset H$ *with* $\chi(f) = p_0$ *such that for* $g \in B$*, the solution* $x^{\chi(g)}$ *of Eq.(1.1) with initial value* $\chi(g)$ *is periodic (and therefore can be defined on* $\mathbb{R}$*). The minimal period of* $x^{\chi(g)}$ *is in* $(\omega - \nu, \omega + \nu)$*. If* $\varphi \in \{p_0\} + V$ *is the initial segment of any periodic solution of* $\dot{x}(t) = -\mu x(t) + g(x(t-1))$ *for some* $g \in B$ *with minimal period in* $(\omega - \nu, \omega + \nu)$*, then* $\varphi = \chi(g)$*. If* $\|g - f\|_{C_b^1} \to 0$*, then* $\chi(g) \to \chi(f) = p_0$ *in* C*.*

In this paper we are going to set $D = \mathbb{R}$.

A discrete Lyapunov functional. As Mallet-Paret and Sell in [21], we define a discrete Lyapunov functional $V : C \setminus \{0\} \to 2\mathbb{N} \cup \{\infty\}$. For $\varphi \in C \setminus \{0\}$, set $sc(\varphi) = 0$ if $\varphi \geq 0$ or $\varphi \leq 0$, otherwise define

$$sc(\varphi) = \sup\Big\{k \in \mathbb{N} \setminus \{0\} : \text{ there exist a strictly increasing sequence } (s_i)_0^k \subseteq [-1, 0] \text{ with } \varphi(s_{i-1})\varphi(s_i) < 0 \text{ for } i \in \{1, 2, .., k\}\Big\}.$$

Then set

$$V(\varphi) = \begin{cases} sc(\varphi), & \text{if } sc(\varphi) \text{ is even or } \infty, \\ sc(\varphi) + 1, & \text{if } sc(\varphi) \text{ is odd.} \end{cases}$$

Also define

$$\begin{aligned} R = \{&\varphi \in C^1 : \varphi(0) \neq 0 \text{ or } \dot{\varphi}(0)\varphi(-1) > 0, \\ &\varphi(-1) \neq 0 \text{ or } \dot{\varphi}(-1)\varphi(0) < 0, \text{ all zeros of } \varphi \text{ are simple}\}. \end{aligned}$$

The linear map $\pi : C \to \mathbb{R}^2$ is given by $\pi(\varphi) = (\varphi(0), \varphi(-1))$.

V has the following lower semi-continuity and continuity property (for a proof, see [17, 22]).

Lemma 2.2. *For each* $\varphi \in C \setminus \{0\}$ *and* $(\varphi_n)_0^\infty \subset C \setminus \{0\}$ *with* $\varphi_n \to \varphi$ *as* $n \to \infty$*,* $V(\varphi) \leq \liminf_{n\to\infty} V(\varphi_n)$*. For each* $\varphi \in R$ *and* $(\varphi_n)_0^\infty \subset C^1 \setminus \{0\}$ *with* $\|\varphi_n - \varphi\|_{C^1} \to 0$ *as* $n \to \infty$*,* $V(\varphi) = \lim_{n\to\infty} V(\varphi_n) < \infty$*.*

The next result explains why V is called a Lyapunov functional.

Lemma 2.3. *Assume that* $\mu \in \mathbb{R}$*,* $J \subset \mathbb{R}$ *is an interval,* $\beta : J \to \mathbb{R}$ *is nonnegative,* $z : J + [-1, 0] \to \mathbb{R}$ *is continuous, and* z *is differentiable on* J*. Suppose that*

$$\dot{z}(t) = -\mu z(t) + \beta(t) z(t-1) \tag{2.2}$$

holds for all $t > \inf J$ *in* J*. Then the following statements hold.*
(i) If $t_1, t_2 \in J$ *with* $t_1 < t_2$ *and* $z_{t_2} \neq 0$*, then* $V(z_{t_1}) \geq V(z_{t_2})$*.*

(ii) If $t, t-2 \in J$, $z(t-1) = z(t) = 0$ *but* $z_t \neq 0$, *then either* $V(z_t) = \infty$ *or* $V(z_{t-2}) > V(z_t)$.
(iii) If β *is positive on* J, $t \in J$, $t-3 \in J$, $z(t) \neq 0$ *for some* $t \in J + [-1,0]$ *and* $V(z_{t-3}) = V(z_t) < \infty$, *then* $z_t \in R$.
(iv) If $J = \mathbb{R}$, β *is bounded and measurable,* z *is bounded and* $z_t \neq 0$ *for all* $t \in \mathbb{R}$, *then* $V(z_t) < \infty$ *for all* $t \in \mathbb{R}$.

Proof. For a positive and continuous β, assertions (i), (ii) and (iii) are shown in [17] and [21]. The proof of Lemma VI.2 in [17] can be modified in a straightforward manner to cover our slightly more general case. Therefore the details are omitted here.

Statement (iv) is a corollary of Theorem 2.4 in [21] with $\delta^* = 1$, $N = 0$, $f^0(t,u,v) = -\mu u + \beta(t) v$. Property I of Theorem 2.4 in [21] holds as β is bounded. □

If nonlinearity f is a C^1−smooth, nondecreasing function and $x, \hat{x} : J + [-1,0] \to \mathbb{R}$ are solutions of Eq. (1.1), then Lemma 2.3 (i) and Lemma 2.3 (ii) can be applied for $z = x - \hat{x}$ with the nonnegative continuous function

$$\beta : J \ni t \mapsto \int_0^1 f'(sx(t-1) + (1-s)\hat{x}(t-1))\,\mathrm{d}s \in [0,\infty).$$

In addition, if $f' > 0$ on $\mathbb{R}$, then β is positive, which condition is needed in Lemma 2.3 (iii).

Proposition 2.4. *Assume* $\mu \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ *is nondecreasing, bounded and either it is continuously differentiable on* $\mathbb{R}$ *or there exist* $u_1 < u_2 < \ldots < u_N$ *with* $N \geq 1$ *so that the restrictions of* f *to the intervals* $(-\infty, u_1]$, $[u_1, u_2]$,.., $[u_{N-1}, u_N]$, $[u_N, \infty)$ *are continuously differentiable. Let* $x : \mathbb{R} \to \mathbb{R}$ *and* $\tilde{x} : \mathbb{R} \to \mathbb{R}$ *be different periodic solutions of* (1.1). *Then* $t \mapsto V(x_t - \tilde{x}_t)$ *is finite and constant. Furthermore,* $\pi(x_t - \tilde{x}_t) \neq (0,0)$ *for all* $t \in \mathbb{R}$.

Proof. The difference $z = x - \tilde{x}$ satisfies equation (2.2) with

$$\beta(t) = \begin{cases} \frac{f(x(t-1)) - f(\tilde{x}(t-1))}{x(t-1) - \tilde{x}(t-1)} & \text{if } x(t-1) \neq \tilde{x}(t-1), \\ D^+ f(x(t-1)) & \text{otherwise,} \end{cases}$$

where $D^+ f$ denotes the right hand side derivative of f. Then β is bounded, measurable and nonnegative. Clearly, $z_t \neq 0$ for all $t \in \mathbb{R}$. Lemma 2.3 (iv) implies $V(z_t) < \infty$ for all $t \in \mathbb{R}$.

Let ω and $\tilde{\omega}$ denote the minimal periods of x and $\tilde{x}$, respectively. If $\tilde{\omega} = 0$ or $\omega/\tilde{\omega}$ is rational, then z is periodic. Thus Lemma 2.3 (i) yields that $t \mapsto V(z_t)$ is constant. If $\omega/\tilde{\omega}$ is irrational, then one may choose sequences $(n_l)_1^\infty \subset \mathbb{Z}$ and $(k_l)_1^\infty \subset \mathbb{Z}$ with

$n_l \to \infty$ and $k_l \to \infty$ as $l \to \infty$ so that $n_l\omega/\tilde{\omega} - k_l \to 0$ as $l \to \infty$. Fix $t \in \mathbb{R}$ arbitrarily. As for all $s \in [-1, 0]$,

$$\begin{aligned} z_{t+n_l\omega}(s) &= x_{t+n_l\omega}(s) - \tilde{x}_{t+n_l\omega}(s) = x_t(s) - \tilde{x}_{t+n_l\omega-k_l\tilde{\omega}}(s) \\ &= x(t+s) - \tilde{x}\left(t + \tilde{\omega}\left(n_l\frac{\omega}{\tilde{\omega}} - k_l\right) + s\right), \end{aligned}$$

we see that $z_{t+n_l\omega}(s)$ tends to $z_t(s) = x(t+s) - \tilde{x}(t+s)$ as $l \to \infty$ uniformly in $s \in [-1, 0]$. So Lemma 2.2 implies

$$V(z_t) \leq \liminf_{l\to\infty} V(z_{t+n_l\omega}).$$

As $\mathbb{R} \ni u \mapsto V(z_u) \in 2\mathbb{N} \cup \{\infty\}$ is monotone nonincreasing by Lemma 2.3 (i), we obtain that $V(z_t) = V(z_{t+u})$ for all $u \geq 0$. As t is arbitrary, we conclude that $t \mapsto V(z_t)$ is constant also in this case.

The second statement now follows from Lemma 2.3 (ii). □

Notation regarding periodic solutions. If $\xi \in \mathbb{R}$ is a zero of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$, then we say that a solution $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) oscillates around ξ if the set of zeros of $x - \xi$ is not bounded from above. Solution $x : \mathbb{R} \to \mathbb{R}$ is called slowly oscillatory around ξ if $V\left(x_t - \hat{\xi}\right) = 2$ for each $t \in \mathbb{R}$, where $\hat{\xi}(s) = \xi$, $s \in [-1, 0]$. A slowly oscillatory solution is defined to be slowly oscillatory around 0. We say $x : \mathbb{R} \to \mathbb{R}$ is rapidly oscillatory around ξ if $V\left(x_t - \hat{\xi}\right) \geq 4$ for all $t \in \mathbb{R}$.

Assume $x : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (1.1) with minimal period ω. We say x is of special symmetry if the relation $x(t + \omega/2) = -x(t)$ holds for all $t \in \mathbb{R}$. Set $t_0 < t_1 < t_0 + \omega$ so that $x(t_0) = \min_{t\in\mathbb{R}} x(t)$ and $x(t_1) = \max_{t\in\mathbb{R}} x(t)$. Solution x is said to be of monotone type if x is nondecreasing on $[t_0, t_1]$ and nonincreasing on $[t_1, t_0 + \omega]$.

Assume that 0 is in the range of a periodic solution $x : \mathbb{R} \to \mathbb{R}$ of (1.1). Then x is normalized if $x(-1) = 0$ and for some $\eta > 0$, $x(s) > 0$ for all $s \in (-1, -1 + \eta)$.

In case $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, and $\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$ are five consecutive zeros of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$, a periodic solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) is called a large amplitude periodic solution if $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. A large-amplitude slowly oscillatory periodic solution $x : \mathbb{R} \to \mathbb{R}$ will be abbreviated as an LSOP solution. This definition is modified for the step function

$$f^{K,0}(x) = \begin{cases} -K & \text{if } x < -1, \\ 0 & \text{if } |x| \leq 1, \\ K & \text{if } x > 1 \end{cases}$$

in the following way. Solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) with nonlinearity $f=f^{K,0}$, $K>0$, is a large-amplitude slowly oscillatory periodic (LSOP) solution if x is of monotone type, special symmetry, and the minimal period of x is in the open interval $(1,2)$.

We have the following simple observation.

Proposition 2.5. *Assume $\mu=1$, $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, nondecreasing, and*

$$\xi_{-2}<\xi_{-1}<\xi_0=0<\xi_1<\xi_2$$

are five consecutive zeros of $\xi\mapsto-\xi+f(\xi)$ with $f'(\xi_j)<1<f'(\xi_k)$ for $j\in\{-2,0,2\}$ and $k\in\{-1,1\}$. Suppose $x:\mathbb{R}\to\mathbb{R}$ is a nontrivial periodic solution of Eq. (1.1) with $x_t\in C_{-2,2}$ for all $t\in\mathbb{R}$. Then the following statements hold. If $\max_{t\in\mathbb{R}}x(t)>0$, then $\xi_1<\max_{t\in\mathbb{R}}x(t)<\xi_2$. If $\max_{t\in\mathbb{R}}x(t)<0$, then $\xi_{-1}<\max_{t\in\mathbb{R}}x(t)<0$. If $\min_{t\in\mathbb{R}}x(t)>0$, then $0<\min_{t\in\mathbb{R}}x(t)<\xi_1$. If $min_{t\in\mathbb{R}}x(t)<0$, then $\xi_{-2}<\min_{t\in\mathbb{R}}x(t)<\xi_{-1}$.

Proof. Assume $x:\mathbb{R}\to\mathbb{R}$ is a periodic solution of Eq. (1.1) with $x_t\in C_{-2,2}$ for all $t\in\mathbb{R}$ and $\max_{t\in\mathbb{R}}x(t)>0$. Choose $t^*\in\mathbb{R}$ so that $x(t^*)=\max_{t\in\mathbb{R}}x(t)$. In case $x(t^*)<\xi_1$ use the fact that $f(x)<x$ for $x\in(0,\xi_1)$ to derive that

$$0=\dot{x}(t^*)=-x(t^*)+f(x(t^*-1))\le-x(t^*)+f(x(t^*))<0,$$

a contradiction. If $x(t^*)=\xi_1$, then Proposition 2.4 implies $x(t^*-1)<x(t^*)$. As f is strictly increasing in a neighborhood of ξ_1, we get that

$$0=\dot{x}(t^*)=-x(t^*)+f(x(t^*-1))<-x(t^*)+f(x(t^*))=0,$$

a contradiction. Hence $x(t^*)>\xi_1$. One may deduce that $\max_{t\in\mathbb{R}}x(t)<\xi_2$ in the same way. We leave the verification of the rest of the statements also to the reader. □

Note that the conditions of the previous proposition are fulfilled if $\mu=1$ and (H1) holds for f.

Boundedness. It is a direct consequence of the next proposition that if $f\in C(\mathbb{R},\mathbb{R})$ is bounded with $\sup_{x\in\mathbb{R}}|f(x)|\le M$ and $p:\mathbb{R}\to\mathbb{R}$ is a periodic solution of (1.1) so that 0 is in the range of p, then $\max_{t\in\mathbb{R}}|p(t)|<M/\mu$.

Proposition 2.6. (Boundedness) *If $\mu>0$, $f:\mathbb{R}\to\mathbb{R}$ is continuous, $\sup_{x\in\mathbb{R}}|f(x)|\le M$ and $x:[t_0-1,\infty)\to\mathbb{R}$ is a solution of (1.1) with $x(t_0)=0$, then $|x(t)|<M/\mu$ for all $t>t_0$.*

Proof. Let $u : \mathbb{R} \to \mathbb{R}$ be the solution of the initial value problem

$$\begin{cases} \dot{u}(t) = -\mu u(t) + M, & t \in \mathbb{R}, \\ u(t_0) = 0. \end{cases}$$

Then $u(t) = M\left(1 - e^{-\mu(t-t_0)}\right)/\mu$ for $t \in \mathbb{R}$. Clearly, if $x : [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of (1.1), then $\dot{x}(t) \leq -\mu x(t) + M$, $t \in \mathbb{R}$. In consequence, Corollary 6.2 of Chapter I in [10] implies that for $t > t_0$, $x(t) \leq u(t) < M/\mu$. The lower bound can be verified analogously. □

The global attractor. In the remaining part of this section, assume (H1). As $f'(x) > 0$ for all $x \in \mathbb{R}$, backward continuation is unique, if it exists. Hence $\Phi(t, \cdot) : C \to C$, $t \geq 0$, are injective (see also [17]), and for every $\varphi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0 = \varphi$. Whenever such a solution on $\mathbb{R}$ exists, it is also denoted by x^{φ}.

The maps $\Phi(t, \cdot) : C \to C$, $t \geq 0$, are monotone with respect to the pointwise ordering on C [17, 26]. As a result, the sets

$$C_{-2,2} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\},$$

$$C_{-2,0} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq 0 \text{ for all } s \in [-1, 0]\},$$

$$C_{0,2} = \{\varphi \in C : 0 \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\}$$

are positively invariant under the semiflow Φ.

There exists a global attractor of the semiflow $\Phi|_{[0,\infty)\times C_{-2,2}}$, i. e., a nonempty, compact set $\mathcal{A} \subset C_{-2,2}$, that is invariant in the sense that $\Phi(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$ and that attracts bounded sets in the sense that for every bounded set $B \subset C_{-2,2}$ and for every open set $U \supset \mathcal{A}$, there exists $t \geq 0$ with $\Phi([t, \infty) \times B) \subset U$. Global attractors are uniquely determined ([9]).

It can be shown that

$$\begin{aligned} \mathcal{A} = \{\varphi \in C_{-2,2} :\ & \text{there is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1)} \\ & \text{with } x(\mathbb{R}) \subset [\xi_{-2}, \xi_2] \text{ and } \varphi = x_0\}, \end{aligned}$$

see [16, 20, 24]. The compactness of $\mathcal{A}$, its invariance property and the injectivity of the maps $\Phi(t, \cdot) : C \to C$, $t \geq 0$, combined permit to verify that the map

$$[0, \infty) \times \mathcal{A} \ni (t, \varphi) \mapsto \Phi(t, \varphi) \in \mathcal{A}$$

extends to a continuous flow $\Phi_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \to \mathcal{A}$; for every $\varphi \in \mathcal{A}$ and for all $t \in \mathbb{R}$ we have $\Phi_{\mathcal{A}}(t, \varphi) = x_t$ with a uniquely determined solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) satisfying $x_0 = \varphi$.

Note that we have $\mathcal{A} = \Phi(1, \mathcal{A}) \subset C^1$; $\mathcal{A}$ is a closed subset of C^1. Using the flow $\Phi_{\mathcal{A}}$ and the continuity of the map

$$C \ni \varphi \mapsto \Phi(1, \varphi) \in C^1,$$

one obtains that C and C^1 define the same topology on $\mathcal{A}$.

Analogously, we define $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ as the global attractors of the restrictions $\Phi|_{[0,\infty) \times C_{-2,0}}$ and $\Phi|_{[0,\infty) \times C_{0,2}}$, respectively. Clearly, $\mathcal{A}_{-2,0} \subset C_{-2,0}$ and $\mathcal{A}_{0,2} \subset C_{0,2}$.

Equilibrium points, periodic solutions and homoclinic connections. All equilibrium points $\hat{\xi} \in C$ of Φ are given so that $\hat{\xi}(s) = \xi$, $-1 \leq s \leq 0$, and ξ satisfies $-\mu\xi + f(\xi) = 0$. The smoothness of f implies that each map $\Phi(t, \cdot)$, $t \geq 0$, is continuously differentiable. For an equilibrium point $\hat{\xi} \in C$, the operators $D_2\Phi\left(t, \hat{\xi}\right) : C \to C$, $t \geq 0$, form a strongly continuous semigroup. The spectrum of the generator of the semigroup consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \mu - \alpha e^{-\lambda} = 0$$

with $\alpha = f'(\xi)$. If $\alpha > 0$, then there is exactly one real λ_0 in the spectrum, the rest of the spectrum is a sequence of complex conjugate pairs $\left(\lambda_j, \overline{\lambda_j}\right)_1^\infty$ with

$$\lambda_0 > \mathrm{Re}\lambda_1 > \mathrm{Re}\lambda_2 > ... > \mathrm{Re}\lambda_n > ...$$

and

$$(2j-1)\pi < \mathrm{Im}\lambda_j < 2j\pi \text{ for } 1 \leq j \in \mathbb{N}.$$

If $0 < f'(\xi) < \mu$, then $\lambda_0 < 0$ and $\hat{\xi}$ is stable. If $f'(\xi) > \mu > 0$, then $\lambda_0 > 0$ and $\hat{\xi}$ is unstable. If $\mu > 0$ and

$$f'(\xi) > \frac{\mu}{\cos\theta_\mu} \text{ for } \theta_\mu \in (3\pi/2, 2\pi) \text{ with } \theta_\mu = -\mu \tan\theta_\mu \tag{2.3}$$

holds, then $\mathrm{Re}\lambda_1 > 0$. In this case if $\varphi \in \mathcal{A}$ belongs to the stable set

$$\mathcal{W}^s\left(\hat{\xi}\right) = \left\{\varphi : \ \omega(\varphi) \text{ exists and } \omega(\varphi) = \hat{\xi}\right\}$$

of $\hat{\xi}$ and $\varphi \neq \hat{\xi}$, then $V\left(\varphi - \hat{\xi}\right) > 2$, see Lemma 3.9 in [24] for a proof.

By hypothesis (H1), $\hat{\xi}_{-2}, \hat{\xi}_{-1}, \hat{0}, \hat{\xi}_1, \hat{\xi}_2$ are the only equilibrium points of Φ in $C_{-2,2}$. In addition, $\hat{\xi}_{-2}$, $\hat{0}$ and $\hat{\xi}_2$ are stable, $\hat{\xi}_{-1}$ and $\hat{\xi}_1$ are unstable.

We are going to use the following additional hypothesis.

(H2) For $j \in \{-1, 1\}$ and $\theta \in (3\pi/2, 2\pi)$ with $\theta = -\mu \tan\theta$, the inequality $f'(\xi_j) > \mu/\cos\theta$ holds.

Note that (H2) is simply condition (2.3) for $\xi = \xi_1$ and $\xi = \xi_{-1}$.

It is shown in Chapter 6 of [17] that if (H1) and (H2) hold, then at least one periodic solution appears with the following two properties: it has range in $(0, \xi_2)$, and it is slowly oscillatory around ξ_1. Analogously, there is at least one periodic solution that is slowly oscillatory around ξ_{-1} and has range in $(\xi_{-2}, 0)$. It is not excluded that more solutions exist with the above properties. It can be proved that the minimal periods of these solutions are in $(1, 2)$. The following proposition holds.

Proposition 2.7. *If conditions (H1) and (H2) are satisfied by f, then there exist periodic solutions $x^1 : \mathbb{R} \to \mathbb{R}$ and $x^{-1} : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) oscillating slowly around ξ_1 and ξ_{-1} with ranges in $(0, \xi_2)$ and $(\xi_{-2}, 0)$, respectively, so that the ranges $x^1(\mathbb{R})$ and $x^{-1}(\mathbb{R})$ are maximal in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly around ξ_1 with ranges in $(0, \xi_2)$; and analogously for x^{-1}.*

Proof. From the paper [22] of Mallet-Paret and Sell we know that the map

$$\pi : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$$

takes the nontrivial periodic orbits of Eq. (1.1) into simple closed curves in $\mathbb{R}^2$, and the images of different periodic orbits are disjoint curves in $\mathbb{R}^2$. Hence for two periodic solutions $\hat{x}$ and $\tilde{x}$ of Eq. (1.1) oscillating slowly around ξ_1, either $\{\pi\hat{x}_t : \ t \in \mathbb{R}\}$ belongs to the interior of $\{\pi\tilde{x}_t : \ t \in \mathbb{R}\}$, or vice versa. Hence it is not difficult to see that either $\hat{x}(\mathbb{R}) \supseteq \tilde{x}(\mathbb{R})$ or $\hat{x}(\mathbb{R}) \subseteq \tilde{x}(\mathbb{R})$ follows.

Suppose for contradiction that there is no periodic solution oscillating slowly around ξ_1 with the stated properties. Then there exists a sequence of periodic solutions $x^n : \mathbb{R} \to \mathbb{R}$ of (1.1) with minimal period $\omega_n \in (1, 2)$, $1 \leq n \in \mathbb{N}$, so that x^n is slowly oscillatory around ξ_1, $x^n(\mathbb{R}) \subseteq x^{n+1}(\mathbb{R}) \subset (0, \xi_2)$ for $n \geq 1$, and there exists no solution $x : \mathbb{R} \to \mathbb{R}$ oscillating slowly around ξ_1 with $x^n(\mathbb{R}) \subseteq x(\mathbb{R}) \subset (0, \xi_2)$ for each $n \geq 1$.

As $x^n(t) \in (0, \xi_2)$ for all $t \in \mathbb{R}$ and f is bounded on $(0, \xi_2)$, Eq. (1.1) gives a uniform upper bound for $|\dot{x}^n|$ on $\mathbb{R}$, $n \geq 1$. Applying the Arzelà–Ascoli theorem and choosing a subsequence if necessary, we obtain that there exist $\omega_* \in [1, 2]$ and a continuous function $x^* : \mathbb{R} \to \mathbb{R}$ such that $\omega_n \to \omega_*$ and x^n converges to x^* as $n \to \infty$ uniformly on each compact subset of the real line. It is easy to see that x^* is periodic with period ω_*. Also, we find that

$$\dot{x}^n(t) \to -\mu x^*(t) + f(x^*(t-1)) \text{ as } n \to \infty$$

uniformly on each compact subinterval of the real line. It follows that x^* is differentiable and satisfies Eq. (1.1) for all $t \in \mathbb{R}$.

As $x^n(\mathbb{R}) \subseteq x^{n+1}(\mathbb{R}) \subset (0, \xi_2)$ for all $n \geq 1$, necessarily

$$0 \leq \min_{t\in\mathbb{R}} x^*(t) \leq \min_{t\in\mathbb{R}} x^n(t) < \xi_1 < \max_{t\in\mathbb{R}} x^n(t) \leq \max_{t\in\mathbb{R}} x^*(t) \leq \xi_2$$

for all $n \geq 1$. We claim that $\min_{t\in\mathbb{R}} x^*(t) > 0$. Indeed, if $t_{\min} \in \mathbb{R}$ is chosen so that $x^*(t_{\min}) = \min_{t\in\mathbb{R}} x^*(t) = 0$, then

$$f(x^*(t_{min} - 1)) = \dot{x}^*(t_{\min}) + \mu x^*(t_{\min}) = 0$$

and hypothesis (H1) implies $x^*(t_{min} - 1) = 0$, a contradiction to Lemma 2.3 (ii), (iv) and the periodicity of x^*. Similarly, $\max_{t\in\mathbb{R}} x^*(t) < \xi_2$.

Proposition 2.4 implies $t \mapsto V\left(x_t^* - \hat{\xi}_1\right)$ is finite and constant. It follows from Lemma 2.2 that

$$V\left(x_t^* - \hat{\xi}_1\right) \leq \liminf_{n\to\infty} V\left(x_t^n - \hat{\xi}_1\right) = 2$$

for all $t \in \mathbb{R}$ and $n \geq 1$. However, $V\left(x_t^* - \hat{\xi}_1\right) > 0$ as function $x^* - \xi_1$ has sign changes. So $V\left(x_t^* - \hat{\xi}_1\right) = 2$ for all $t \in \mathbb{R}$.

We conclude that solution x^* is periodic, slowly oscillatory around ξ_1, has range in $(0, \xi_2)$, and $x^n(\mathbb{R}) \subseteq x(\mathbb{R}) \subset (0, \xi_2)$ for each $n \geq 1$, a contradiction to our initial assumption.

The proof is analogous for x^{-1}. □

Note that under hypothesis (H1) there is no homoclinic orbit to $\hat{\xi}_j$, $j \in \{-2, 0, 2\}$ as they are stable. It follows from Proposition 3.1 in [15], that there exit no homoclinic orbits to $\hat{\xi}_i$, $i \in \{-1, 1\}$.

The Poincaré–Bendixson Theorem. Suppose that f satisfies (H1). If $\varphi \in C_{-2.2}$, then $\omega(\varphi)$ is either a single nonconstant periodic orbit or for each $\psi \in \omega(\varphi)$,

$$\alpha(\psi) \cup \omega(\psi) \subseteq \left\{\hat{\xi}_{-2}, \hat{\xi}_{-1}, \hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\right\},$$

see [22]. An analogous result holds for $\alpha(x)$ in case x is defined on $\mathbb{R}$ and $\{x_t : t \leq 0\} \subset C_{-2,2}$.

3. LSOP Solutions For Special Nonlinearities

In the remaining part of the paper we fix $\mu = 1$. The results can be easily modified for different values of $\mu > 0$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a C^∞-smooth function such that $\rho(t) = 0$ for $t \leq 0$, $\rho(t) = 1$ for $t \geq 1$ and $\rho'(t) > 0$ for $t \in (0, 1)$. For given $K > 0$ and $\varepsilon \in (0, 1)$, define $f^{K,\varepsilon} : \mathbb{R} \to \mathbb{R}$

(Fig. 4) by

$$f^{K,\varepsilon}(x) = K\rho\left(\frac{|x|-1}{\varepsilon}\right)\operatorname{sgn}(x).$$

The function $f^{K,0} : \mathbb{R} \to \mathbb{R}$ (Fig. 4) is given by

$$f^{K,0}(x) = \lim_{\varepsilon\to 0+} f^{K,\varepsilon}(x) = \begin{cases} -K & \text{if } x < -1, \\ 0 & \text{if } |x| \le 1, \\ K & \text{if } x > 1. \end{cases}$$

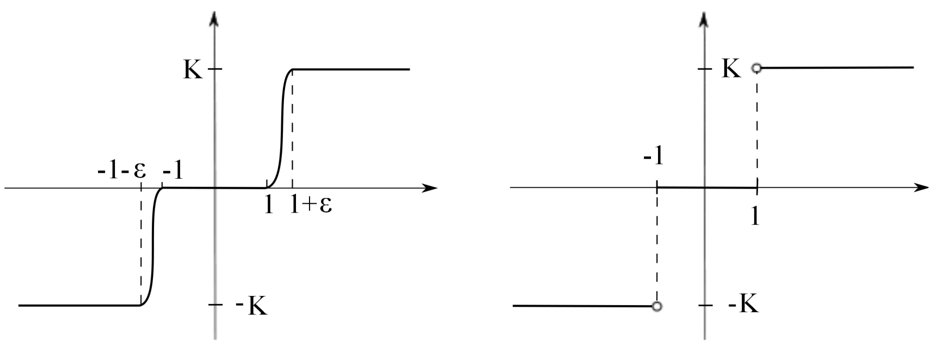

FIGURE 4. The plot of $f^{K,\varepsilon}$ for $\varepsilon > 0$ small and for $\varepsilon = 0$

Note that for all $\varepsilon \in (0,1)$ and $K > 1+\varepsilon$, the function $\xi \mapsto -\xi + f^{K,\varepsilon}(\xi)$ admits exactly five zeros

$$\xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2$$

with $\xi_{-2} = -K$, $\xi_{-1} \in (-1-\varepsilon, -1)$, $\xi_0 = 0$, $\xi_1 \in (1, 1+\varepsilon)$ and $\xi_2 = K$.

Consider the delay differential equation

$$\dot{x}(t) = -x(t) + f^{K,\varepsilon}(x(t-1)). \tag{3.1}$$

Set $J_i^\varepsilon = \left(f^{K,\varepsilon}\right)^{-1}(i)$ for $i \in \{-K, 0, K\}$.

If $t_0 < t_1$ and $x : [t_0 - 1, t_1] \to \mathbb{R}$ is a solution of Eq. (3.1) such that for some $i \in \{-K, 0, K\}$, we have $x(t-1) \in J_i^\varepsilon$ for all $t \in (t_0, t_1)$, then Eq. (3.1) reduces to the ordinary differential equation

$$\dot{x}(t) = -x(t) + i$$

on the interval (t_0, t_1), and thus

$$x(t) = i + (x(t_0) - i)\, e^{-(t-t_0)} \qquad \text{for } t \in [t_0, t_1]. \tag{3.2}$$

We say that a function $x : [t_0, t_1] \to \mathbb{R}$ is *of type* (i) *on* $[t_0, t_1]$ for some $i \in \{-K, 0, K\}$ if (3.2) holds. If $x : [t_0 - 1, t_1] \to \mathbb{R}$ is a solution of Eq. (3.1) so that x is type of (i) on $[t_0 - 1, t_1 - 1]$ for some $i \in \{-K, 0, K\}$, then with $j = x(t_0 - 1)$ the equality

$$x(t) = x(t_0) e^{-(t-t_0)} + \int_0^{t-t_0} e^{-(t-t_0-s)} f^{K,\varepsilon}\left(i + (j-i) e^{-s}\right) ds \tag{3.3}$$

holds for all $t \in [t_0, t_1]$. This motivates the next definition. A function $x : [t_0, t_1] \to \mathbb{R}$ is *of type* (i, j) *on* $[t_0, t_1]$ with $i \in \{-K, 0, K\}$ and $j \in \mathbb{R}$ if (3.3) holds for all $t \in [t_0, t_1]$.

In the rest of the section assume that $K > 3$.

Let

$$T(\varepsilon) = \ln(1+\varepsilon),\ \hat{T}(\varepsilon) = \ln \frac{K-1}{K-1-\varepsilon},\ \tilde{T}(\varepsilon) = \ln \frac{K+1+\varepsilon}{K+1}$$

denote the times that a function of type (0) needs to decrease from $1+\varepsilon$ to 1 or to increase from $-1-\varepsilon$ to -1, a function of type $(-K)$ needs to decrease from -1 to $-(1+\varepsilon)$, a function of type $(-K)$ needs to decrease from $1+\varepsilon$ to 1, respectively. Clearly, $T(0) = \hat{T}(0) = \tilde{T}(0) = 0$.

3.1. **An LSOP solution for nonlinearity $f^{K,\varepsilon}$.** Define

$$U^1 = \left\{(a,\varepsilon) \in (0,1)^3 \times [0,1) :\ a = (a_1, a_2, a_3),\ a_1 + a_2 + a_3 + 2T(\varepsilon) + \hat{T}(\varepsilon) < 1\right\}.$$

It is easy to see that U^1 is open in $(0,1)^3 \times [0,1)$.

For given $(a, \varepsilon) \in U^1$, set

$$\begin{aligned}
s_0 &= -1,\\
s_1 &= s_0 + a_1 = -1 + a_1,\\
s_1^* &= s_1 + T(\varepsilon) = -1 + a_1 + T(\varepsilon),\\
s_2 &= s_1^* + a_2 = -1 + a_1 + T(\varepsilon) + a_2,\\
s_2^* &= s_2 + \hat{T}(\varepsilon) = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon),\\
s_3 &= s_2^* + a_3 = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3,\\
s_3^* &= s_3 + T(\varepsilon) = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3 + T(\varepsilon).
\end{aligned}$$

Clearly $s_i = s_i^*$, $i \in \{1, 2, 3\}$, for $\varepsilon = 0$.

Define $h = h(a,\varepsilon) : \mathbb{R} \to \mathbb{R}$ (Fig. 5) by

$$h(t) = \begin{cases} K & \text{if } t < s_1, \\ f^{K,\varepsilon}\left((1+\varepsilon)\, e^{-(t-s_1)}\right) & \text{if } s_1 \leq t < s_1^*, \\ 0 & \text{if } s_1^* \leq t < s_2, \\ f^{K,\varepsilon}\left(-K + (K-1)\, e^{-(t-s_2)}\right) & \text{if } s_2 \leq t < s_2^*, \\ -K & \text{if } s_2^* \leq t < s_3, \\ f^{K,\varepsilon}\left(-(1+\varepsilon)\, e^{-(t-s_3)}\right) & \text{if } s_3 \leq t < s_3^*, \\ 0 & \text{if } s_3^* \leq t. \end{cases}$$

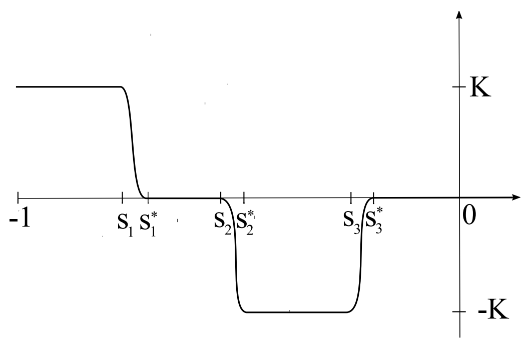

FIGURE 5. The plot of $h(a,\varepsilon)$

Define the map $\Sigma : U^1 \to C$ by

$$\Sigma(a,\varepsilon)(t) = e^{-t} \int_{-1}^{t} e^s h(a,\varepsilon)(s)\, ds \qquad (-1 \leq t \leq 0)\,. \tag{3.4}$$

We look for initial segments of LSOP solutions in the set $\Sigma(U^1) \subset C$.

Notice that $\Sigma(a,\varepsilon)$ is the unique solution of the initial value problem

$$\begin{cases} \dot{y}(t) = -y(t) + h(a,\varepsilon)(t) & (-1 \leq t \leq 0) \\ y(-1) = 0. \end{cases} \tag{3.5}$$

Proposition 3.1. *$\Sigma : U^1 \to C$ is continuous.*

Proof. The continuity of the map

$$U^1 \ni (a,\varepsilon) \mapsto h(a,\varepsilon)|_{[-1,0]} \in L^1(0,1)$$

follows in a straightforward way from the definition of $h(a,\varepsilon)$. Applying formula (3.4), the continuity of Σ is obvious. □

For each fixed $(a,\varepsilon) \in U^1 \cap (0,1)^3 \times (0,1)$, the map $[-1,0] \ni t \mapsto h(a,\varepsilon)(t) \in \mathbb{R}$ is C^1-smooth with derivative $h'(a,\varepsilon)(t)$.

For given $\varepsilon \in [0,1)$, define

$$U_\varepsilon^1 = \left\{ a \in (0,1)^3 : (a,\varepsilon) \in U^1 \right\}.$$

The definition of U^1 implies that U_ε^1 is open.

If $a \in U_\varepsilon^1$ and $|\delta| < \frac{1}{2}\min\{a_1,a_2,a_3\}$, then

$$h(a_1+\delta,a_2,a_3,\varepsilon)(t) = h(a,\varepsilon)(t-\delta) \quad \text{for } t \in [-1,0],$$

$$h(a_1,a_2+\delta,a_3,\varepsilon)(t) = \begin{cases} h(a,\varepsilon)(t) & \text{for } t \in \left[-1, s_1^* + \frac{a_2}{2}\right], \\ h(a,\varepsilon)(t-\delta) & \text{for } t \in \left[s_1^* + \frac{a_2}{2}, 0\right], \end{cases}$$

$$h(a_1,a_2,a_3+\delta,\varepsilon)(t) = \begin{cases} h(a,\varepsilon)(t) & \text{for } t \in \left[-1, s_2^* + \frac{a_3}{2}\right], \\ h(a,\varepsilon)(t-\delta) & \text{for } t \in \left[s_2^* + \frac{a_3}{2}, 0\right]. \end{cases}$$

Now it is clear that we have

$$\frac{\partial}{\partial a_i} h(a,\varepsilon)(t) = \begin{cases} 0 & \text{for } t \in [-1,s_i] \\ -h'(a,\varepsilon)(t) & \text{for } t \in [s_i,0] \end{cases}$$

for $i \in \{1,2,3\}$. Define $\psi_i \in C$, $i \in \{1,2,3\}$, by

$$\psi_i(t) = \psi_i(a,\varepsilon)(t) = e^{-t} \int_{-1}^{t} e^s \frac{\partial}{\partial a_i} h(a,\varepsilon)(s)\, ds \qquad (t \in [-1,0]).$$

Obviously ψ_1, ψ_2 and ψ_3 are linearly independent elements of C. With the above notation, we obtain the following C^1-smoothness property of Σ.

Proposition 3.2. *For each fixed $\varepsilon \in (0,1)$, the map $U_\varepsilon^1 \ni a \mapsto \Sigma(a,\varepsilon) \in C$ is C^1-smooth with $D_a\Sigma(a,\varepsilon)(b) = b_1\psi_1 + b_2\psi_2 + b_3\psi_3$ for all $a \in U_\varepsilon^1$ and $b = (b_1,b_2,b_3) \in \mathbb{R}^3$.*

Proof. $\Sigma(a,\varepsilon)$ is the unique solution of the initial value problem (3.5). Hence the claim of the proposition follows from the differentiability of solutions of ordinary differential equations with respect to the parameters. □

Let

$$\begin{aligned} U^2 = \big\{ (a,\varepsilon) \in U^1 : & \ \Sigma(a,\varepsilon)(s) > 1+\varepsilon \text{ for } s \in [s_1,s_1^*], \\ & |\Sigma(a,\varepsilon)(s)| < 1 \text{ for } s \in [s_2,s_2^*], \\ & \Sigma(a,\varepsilon)(s) < -1-\varepsilon \text{ for } s \in [s_3,s_3^*] \big\}. \end{aligned}$$

Proposition 3.1 and the definition of U^2 imply that U^2 is open in $(0,1)^3 \times [0,1)$.

For $(a,\varepsilon)\in U^2$, consider the solution $x=x^{\Sigma(a,\varepsilon)}:[-1,\infty)\to\mathbb{R}$ of Eq. (3.1).

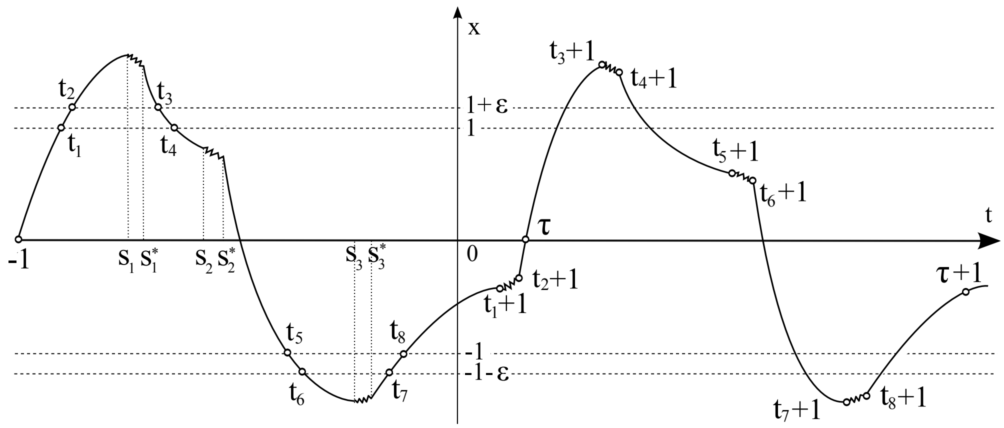

FIGURE 6. Solution $x^{\Sigma(a,\varepsilon)}$ of Eq. (3.1)

By the definition of U^2, there exist $t_1,t_2,...,t_6$ in $[-1,0]$ such that

$$-1<t_1\le t_2<s_1\le s_1^*<t_3\le t_4<s_2\le s_2^*<t_5\le t_6<s_3\le s_3^*$$

and

$$x(t_1)=1,\ x(t_2)=1+\varepsilon,\ x(t_3)=1+\varepsilon,\ x(t_4)=1,\ x(t_5)=-1,\ x(t_6)=-1-\varepsilon$$

(see Fig. 6).

For $\varepsilon\in(0,1)$, introduce

$$c_1(\varepsilon)=\int_0^{T(\varepsilon)} e^s f^{K,\varepsilon}\left((1+\varepsilon)e^{-s}\right)ds,\quad c_2(\varepsilon)=\int_0^{\hat{T}(\varepsilon)} e^s f^{K,\varepsilon}\left(K-(K-1)e^{-s}\right)ds.$$

These integrals appear in the explicit evaluation of a return map. Observe that

$$c_1(\varepsilon)=\int_0^1 \frac{\varepsilon(1+\varepsilon)}{(1+\varepsilon u)^2}K\rho(u)\,du\qquad(\varepsilon\in(0,1)),$$

$$c_2(\varepsilon)=\int_0^1 \frac{\varepsilon}{(K-1-\varepsilon u)^2}K(K-1)\rho(u)\,du\qquad(\varepsilon\in(0,1)).$$

From the last two equalities it is elementary to show that with the extension $c_1(0)=0$, $c_2(0)=0$ of c_1, c_2 from $(0,1)$ to $[0,1)$, the functions c_1 and c_2 are C^1-smooth on $[0,1)$.

We also need the following integrals:

$$
\begin{aligned}
I_1 &= \int_{-1}^{s_1} e^s h(a,\varepsilon)(s)\,ds = K\left(e^{s_1} - e^{-1}\right) = \frac{K}{e}\left(e^{a_1} - 1\right), \\
I_{1,*} &= \int_{-1}^{s_1^*} e^s h(a,\varepsilon)(s)\,ds = I_1 + \int_{s_1}^{s_1^*} e^s f^{K,\varepsilon}\left((1+\varepsilon)\,e^{-(s-s_1)}\right) ds \\
&= I_1 + e^{s_1} c_1(\varepsilon) = \frac{1}{e}\left[K\left(e^{a_1} - 1\right) + e^{a_1} c_1(\varepsilon)\right], \\
I_2 &= \int_{-1}^{s_2} e^s h(a,\varepsilon)(s)\,ds = I_{1,*}, \\
I_{2,*} &= \int_{-1}^{s_2^*} e^s h(a,\varepsilon)(s)\,ds = I_2 + \int_{s_2}^{s_2^*} e^s f^{K,\varepsilon}\left(-K + (K-1)\,e^{-(s-s_2)}\right) ds \\
&= I_2 - e^{s_2} c_2(\varepsilon) \\
&= \frac{1}{e}\left[K\left(e^{a_1} - 1\right) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)\,c_2(\varepsilon)\right], \\
I_3 &= \int_{-1}^{s_3} e^s h(a,\varepsilon)(s)\,ds = I_{2,*} + \int_{s_2^*}^{s_3} e^s(-K)\,ds = I_{2,*} + Ke^{s_2^*} - Ke^{s_3} \\
&= \frac{1}{e}\Bigg[K\left(e^{a_1} - 1\right) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)\,c_2(\varepsilon) \\
&\quad + e^{a_1+a_2}\left(1 - e^{a_3}\right)\frac{(1+\varepsilon)\,K\,(K-1)}{K-1-\varepsilon}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
I_{3,*} &= \int_{-1}^{s_3^*} e^s h(a,\varepsilon)(s)\,ds = I_3 + \int_{s_3}^{s_3^*} e^s f^{K,\varepsilon}\left(-(1+\varepsilon)\,e^{-(s-s_3)}\right) ds \\
&= I_3 - e^{s_3} c_1(\varepsilon) \\
&= \frac{1}{e}\Bigg[K\left(e^{a_1} - 1\right) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)\,c_2(\varepsilon) \\
&\quad + e^{a_1+a_2}\left(1 - e^{a_3} - e^{a_3} c_1(\varepsilon)\right)\frac{(1+\varepsilon)\,K\,(K-1)}{K-1-\varepsilon}\Bigg].
\end{aligned}
$$

Notice that $I_1, I_{1,*}, ..., I_3, I_{3,*}$ are C^1-smooth functions from U^2 into $\mathbb{R}$, and

$$
e^{-s_j} I_j = \Sigma(a,\varepsilon)(s_j), \quad e^{-s_j^*} I_{j,*} = \Sigma(a,\varepsilon)\left(s_j^*\right)
$$

for all $j \in \{1,2,3\}$.

For t_1 and t_2,

$$
e^{-t_1}\int_{-1}^{t_1} Ke^s ds = 1 \text{ and } e^{-t_2}\int_{-1}^{t_2} Ke^s ds = 1 + \varepsilon
$$

hold, respectively. Hence

$$t_1 = \ln \frac{K}{K-1} - 1,\; t_2 = \ln \frac{K}{K-1-\varepsilon} - 1 \text{ and } t_2 - t_1 = \ln \frac{K-1}{K-1-\varepsilon} = \hat{T}(\varepsilon)\,. \tag{3.6}$$

Proposition 3.3. *The maps*

$$U^2 \ni (a,\varepsilon) \mapsto x^{\Sigma(a,\varepsilon)}(t_1+1) = \frac{K-1}{K} I_{3,*} \in \mathbb{R},$$

$$U^2 \ni (a,\varepsilon) \mapsto x^{\Sigma(a,\varepsilon)}(t_2+1) = \frac{K-1-\varepsilon}{K} I_{3,*} + \frac{K-1-\varepsilon}{K-1} c_2(\varepsilon) \in \mathbb{R},$$

$$U^2 \ni (a,\varepsilon) \mapsto x^{\Sigma(a,\varepsilon)}(t_3+1) = K + \frac{K(1+\varepsilon)}{e(K-1-\varepsilon) I_{1,*}} \left(x^{\Sigma(a,\varepsilon)}(t_2+1) - K \right) \in \mathbb{R}$$

are continuously differentiable.

Proof. Since $I_{3,*}$, c_2, $I_{1,*}$ are C^1-smooth functions on U^2, one has to show only the stated equalities for $x^{\Sigma(a,\varepsilon)}(t_i+1)$, $i \in \{1,2,3\}$. Set $x = x^{\Sigma(a,\varepsilon)}$.

From $x(s) \in [0,1]$, $-1 \le s \le t_1$, it follows that x is of type (0) on $[0, t_1+1]$. The definition of $\Sigma(a,\varepsilon)$ gives that x is of type (0) on $[s_3^*, 0]$ as well. Then

$$x(t) = e^{-\left(t-s_3^*\right)} x(s_3^*) \qquad (s_3^* \le t \le t_1+1)\,, \tag{3.7}$$

and using (3.4), (3.6) and the definitions of $I_{3,*}$ and $c_2(\varepsilon)$, we get

$$x(t_1+1) = e^{-(t_1+1)} e^{s_3^*} x(s_3^*) = \frac{K-1}{K} I_{3,*}$$

and

$$\begin{aligned} x(t_2+1) &= e^{t_1-t_2} x(t_1+1) + e^{t_1-t_2} \int_0^{t_2-t_1} e^s f^{K,\varepsilon}\left(K - (K-1)e^{-s}\right) ds \\ &= \frac{K-1-\varepsilon}{K} I_{3,*} + \frac{K-1-\varepsilon}{K-1} c_2(\varepsilon)\,. \end{aligned}$$

As x is of type (K) on $[t_2+1, t_3+1]$, we find that

$$x(t_3+1) = e^{t_2-t_3}\left(x(t_2+1) - K\right) + K. \tag{3.8}$$

From $s_1^* < t_3 < s_2$, (3.4) and $h(a,\varepsilon)(t) = 0$ for $t \in [s_1^*, t_3]$, $x(t_3) = e^{-t_3} I_{1,*}$ follows. Since $x(t_3) = 1+\varepsilon$, one concludes that

$$t_3 = \ln \frac{I_{1,*}}{1+\varepsilon}. \tag{3.9}$$

Substituting t_2 and t_3 from (3.6) and (3.9) into (3.8), the proof is complete. □

Now we are in a position to define a further proper subset of U^1. Let

$$U^3 = \left\{(a,\varepsilon) \in U^2 : \ x^{\Sigma(a,\varepsilon)}(t_1+1) > -1, \ x^{\Sigma(a,\varepsilon)}(t_2+1) < 0, \ x^{\Sigma(a,\varepsilon)}(t_3+1) > 0\right\}.$$

At this stage we do not know whether $U^3 \neq 0$. However, Proposition 3.3 and the definition of U^3 imply that U^3 is open in $(0,1)^3 \times [0,1)$. A typical element of $\Sigma(U^3)$ is presented in Fig. 6.

The next remark plays a prominent role in proving Theorem 1.1, as well as Remark 3.14 of the next subsection.

Remark 3.4. Observe that any $\varphi \in \Sigma(U^3)$ can be characterized as follows: there exist $\varepsilon \in [0,1)$ and

$$-1 < s_1 \leq s_1^* < s_2 \leq s_2^* < s_3 \leq s_3^* < 0$$

with

$$s_1^* - s_1 = T(\varepsilon), \ s_2^* - s_2 = \hat{T}(\varepsilon), \ s_3^* - s_3 = T(\varepsilon)$$

so that $\varphi \in C$ satisfies

(i) $\varphi(-1) = 0$,
(ii) φ is of type (K) on $[-1, s_1]$,
(iii) φ is of type $(0, 1+\varepsilon)$ on $[s_1, s_1^*]$,
(iv) φ is of type (0) on $[s_1^*, s_2]$,
(v) φ is of type $(-K, -1)$ on $[s_2, s_2^*]$,
(vi) φ is of type $(-K)$ on $[s_2^*, s_3]$,
(vii) φ is of type $(0, -1-\varepsilon)$ on $[s_3, s_3^*]$,
(viii) φ is of type (0) on $[s_3^*, 0]$,
(ix) $\varphi(s) > 1+\varepsilon$ for $s \in [s_1, s_1^*]$,
(x) $|\varphi(s)| < 1$ for $s \in [s_2, s_2^*]$,
(xi) $\varphi(s) < -1-\varepsilon$ for $s \in [s_3, s_3^*]$,
(xii) if $-1 < t_1 < s_1$ with $\varphi(t_1) = 1$, then $x^{\varphi}(t_1+1) > -1$,
(xiii) if $t_1 \leq t_2 < s_1$ with $\varphi(t_2) = 1+\varepsilon$, then $x^{\varphi}(t_2+1) < 0$,
(xiv) if $s_1^* < t_3 < s_2$ with $\varphi(t_3) = 1+\varepsilon$, then $x^{\varphi}(t_3+1) > 0$.

Notice that (i)-(viii) characterize $\varphi \in \Sigma(U^1)$, and (i)-(xi) characterize $\varphi \in \Sigma(U^2)$.

If $(a,\varepsilon) \in U^3$, then for $x = x^{\Sigma(a,\varepsilon)}$ we have $x(s_3^*) < -1-\varepsilon$, x is of type (0) on $[s_3^*, t_1+1]$ and $x(t_1+1) > -1$. So t_7 and t_8 can be uniquely defined by

$$s_3^* < t_7 \leq t_8 < t_1+1, \quad x(t_7) = -1-\varepsilon, \quad x(t_8) = -1.$$

In addition, from $(a,\varepsilon)\in U^3$ it follows that x has a zero in (t_2+1,t_3+1). Since x is of type (K) on $[t_2+1,t_3+1]$, there is a unique zero. Let τ denote the zero of $x^{\Sigma(a,\varepsilon)}$ in (t_2+1,t_3+1) (Fig. 6).

Proposition 3.5. *Suppose $(a,\varepsilon)\in U^3$ and define $t_1,t_2,...,t_8$ and τ for $x=x^{\Sigma(a,\varepsilon)}$ as above. Then $x_{\tau+1}\in\Sigma\left(U^1\right)$ and*

$$x_{\tau+1}=\Sigma\left(t_3+1-\tau,t_5-t_4,t_7-t_6,\varepsilon\right).$$

Proof. Notice that τ is the first positive zero of x. Indeed, we know that the function x strictly increases on $[s_3^*,t_1+1]$ from $x\left(s_3^*\right)<-1-\varepsilon$ to $x\left(t_1+1\right)\in(-1,0)$ and strictly increases on $[t_2+1,t_3+1]$ from $x\left(t_2+1\right)<0$ to $x\left(t_3+1\right)>0$. It remains to consider x on $[t_1+1,t_2+1]$, where it is of type $(K,1)$, that is

$$x\left(t\right)=e^{-(t-t_1-1)}x\left(t_1+1\right)+\int_0^{t-t_1-1}e^{-(t-t_1-1-s)}f^{K,\varepsilon}\left(K+\left(1-K\right)e^{-s}\right)ds \tag{3.10}$$

for $t_1+1\leq t\leq t_2+1$. The case $\varepsilon=0$ is evident. If $\varepsilon>0$ and $z\in(t_1+1,t_2+1)$ is any zero of x, then

$$\dot{x}\left(z\right)=f^{K,\varepsilon}\left(x\left(z-1\right)\right)=f^{K,\varepsilon}\left(K-Ke^{-z}\right)>f^{K,\varepsilon}\left(K-Ke^{-t_1-1}\right)=f^{K,\varepsilon}\left(1\right)=0.$$

Hence it is easy to see that the existence of a zero of x in (t_1+1,t_2+1) implies $x\left(t_2+1\right)>0$, a contradiction. Thus $x\left(t\right)<0$ follows for all $t\in[0,\tau)$.

From (3.10) one easily obtains that $x\left(t_1+1\right)\leq x\left(t\right)$ for $t\in[t_1+1,t_2+1]$.

Now it should be clear that

$x\left(\tau\right)=0$,
x is of type (K) on $[\tau,t_3+1]$,
x is of type $(0,1+\varepsilon)$ on $[t_3+1,t_4+1]$,
x is of type (0) on $[t_4+1,t_5+1]$,
x is of type $(-K,-1)$ on $[t_5+1,t_6+1]$,
x is of type $(-K)$ on $[t_6+1,t_7+1]$,
x is of type $(0,-1-\varepsilon)$ on $[t_7+1,t_8+1]$,
x is of type (0) on $[t_8+1,\tau+1]$.

It remains to show that

$$t_4-t_3=T\left(\varepsilon\right),\quad t_6-t_5=\hat{T}\left(\varepsilon\right),\quad t_8-t_7=T\left(\varepsilon\right),$$

which relations are consequences of the definitions of t_3,t_4,t_5,t_6,t_7,t_8, $T\left(\varepsilon\right)$, $\hat{T}\left(\varepsilon\right)$ and the facts that x is of type (0) on $[t_3,t_4]$ and on $[t_7,t_8]$ and that x is of type $(-K)$ on $[t_5,t_6]$. The proof is complete. □

We remark that if $x_{\tau+1}^{\Sigma(a,\varepsilon)} = \Sigma(a,\varepsilon)$ holds for some $(a,\varepsilon) \in U^3$, i. e.,

$$a_1 = t_3 + 1 - \tau,\ \ a_2 = t_5 - t_4,\ \ a_3 = t_7 - t_6,$$

then x is a periodic solution of Eq. (3.1) with minimal period $\tau + 1$. The dependence of $t_3 + 1 - \tau,\ t_5 - t_4$ and $t_7 - t_6$ on (a,ε) is considered in the next result.

Proposition 3.6. *Suppose $(a,\varepsilon) \in U^3$ and define t_3, t_4, t_5, t_6, t_7 and τ as in Proposition 3.5. Then*

$$\begin{aligned} t_3 + 1 - \tau &= 1 + \ln \frac{I_{1,*}}{1+\varepsilon} - \ln\left(\frac{K}{K-1-\varepsilon} - \frac{I_{3,*}}{K} - \frac{c_2(\varepsilon)}{K-1}\right), \\ t_5 - t_4 &= \ln \frac{I_{2,*} + Ke^{s_2^*}}{(K-1)\, I_{1,*}}, \\ t_7 - t_6 &= \ln \frac{-I_{3,*}(K-1-\varepsilon)}{(1+\varepsilon)\left(I_{2,*} + Ke^{s_2^*}\right)}. \end{aligned}$$

In particular, if $\varepsilon = 0$, that is $(a,0) \in U^3$, then

$$\begin{aligned} t_3 + 1 - \tau &= a_1 + \ln \frac{K(K-1)\left(1-e^{-a_1}\right)}{K + (K-1)\, e^{-1}\left(1 + e^{a_1+a_2+a_3} - e^{a_1} - e^{a_1+a_2}\right)}, \\ t_5 - t_4 &= a_2 + \ln \frac{e^{-a_2}\left(1-e^{-a_1}\right)+1}{(K-1)\left(1-e^{-a_1}\right)}, \\ t_7 - t_6 &= a_3 + \ln\left[(K-1)\left(\frac{e^{a_1+a_2}}{e^{a_1} + e^{a_1+a_2} - 1} - e^{-a_3}\right)\right]. \end{aligned}$$

Proof. Applying that x is of type (K) on $[t_2+1, \tau]$, an integration gives

$$0 = e^{\tau} x(\tau) = e^{t_2+1} x(t_2+1) + K\left(e^{\tau} - e^{t_2+1}\right).$$

Hence, using also Proposition 3.3,

$$\tau = \ln\left(\frac{K}{K-1-\varepsilon} - \frac{I_{3,*}}{K} - \frac{c_2(\varepsilon)}{K-1}\right). \tag{3.11}$$

This formula combined with (3.9) yields the result for $t_3 + 1 - \tau$.

It follows from (3.9) and the definition of $T(\varepsilon)$ that $t_4 = \ln I_{1,*}$. Since $s_2^* < t_5 < t_6 < s_3$,

$$\begin{aligned} -1 &= e^{-t_5}\left(I_{2,*} + \int_{s_{2*}}^{t_5} e^s(-K)\, ds\right) = e^{-t_5}\left(I_{2,*} + Ke^{s_2^*}\right) - K, \\ -1-\varepsilon &= e^{-t_6}\left(I_{2,*} + \int_{s_{2*}}^{t_6} e^s(-K)\, ds\right) = e^{-t_6}\left(I_{2,*} + Ke^{s_2^*}\right) - K. \end{aligned}$$

So

$$t_5 = \ln\left(\frac{I_{2,*}}{K-1} + \frac{K}{K-1}e^{s_2^*}\right),$$
$$t_6 = \ln\left(\frac{I_{2,*}}{K-1-\varepsilon} + \frac{K}{K-1-\varepsilon}e^{s_2^*}\right).$$

Using that x is of type (0) on $[s_3^*, t_1+1]$ and $s_3^* < t_7 < t_1+1$,

$$-1-\varepsilon = e^{-\left(t_7 - s_3^*\right)}x\left(s_3^*\right) = e^{-t_7}I_{3,*}$$

and

$$t_7 = \ln\frac{-I_{3,*}}{1+\varepsilon}$$

follow. Therefore

$$t_5 - t_4 = \ln\frac{I_{2,*} + Ke^{s_2^*}}{(K-1)\,I_{1,*}},$$
$$t_7 - t_6 = \ln\frac{-I_{3,*}\,(K-1-\varepsilon)}{(1+\varepsilon)\left(I_{2,*} + Ke^{s_2^*}\right)}.$$

The case $\varepsilon = 0$ is an elementary exercise. □

The above results allow us to define the map $F : U^3 \to \mathbb{R}^3$ by

$$F(a,\varepsilon) = (t_3 + 1 - \tau, t_5 - t_4, t_7 - t_6),$$

where t_3, t_4, t_5, t_6, t_7 and τ are uniquely determined by the solution $x^{\Sigma(a,\varepsilon)}$ of Eq. (3.1). An immediate consequence of the explicit representation of $F(a,\varepsilon)$ in terms of (a,ε) and the C^1-smoothness of the involved functions:

Proposition 3.7. *F is C^1-smooth.*

If $(a,\varepsilon) \in U^3$ and $F(a,\varepsilon) = a$, then $x^{\Sigma(a,\varepsilon)}$ is a periodic solution of Eq. (3.1) with minimal period $\tau + 1$. A first step to find a solution of $F(a,\varepsilon) = a$ in U^3 is to consider the case $\varepsilon = 0$. Set

$$U_0^3 = \left\{a \in \mathbb{R}^3 : (a,0) \in U^3\right\}.$$

Let K^* be the unique solution of $w(K) = 1/e$ on $(3,\infty)$, where

$$w(K) = \frac{\left(K^2 - 2K - 1\right)^2}{(K-1)\,(K+1)^3}.$$

Then K^* is well-defined. Indeed, $w(3) = 1/32$, $\lim_{K\to\infty} w(K) = 1$, and as $K \mapsto 2K/\left(K^2-1\right)$ and $K \mapsto (4K+2)/(K+1)^2$ are strictly decreasing functions on $(3,\infty)$,

$$w(K) = \left(1 - \frac{2K}{K^2-1}\right)\left(1 - \frac{4K+2}{(K+1)^2}\right)$$

is strictly increasing on $(3, \infty)$. Evaluating $w(6)$ and $w(7)$, one sees that $K^* \in (6, 7)$. We have the numerical approximation $K^* \approx 6.8653$. Note that $w(K) > 1/e$ for $K > K^*$.

Proposition 3.8. *For $K \in (3, K^*]$, equation $F(a, 0) = a$ admits no solution in U_0^3. For $K > K^*$, there is a unique $a^* \in U_0^3$ with $F(a^*, 0) = a^*$.*

Proof. Assume $K > 3$. First observe that $a \in \mathbb{R}^3$ is a solution of $F(a, 0) = a$ if and only if

$$a_2 = -\ln\left(K - 1 - \frac{1}{1 - e^{-a_1}}\right), \tag{3.12}$$

$$a_3 = \ln\left((K - 1)\left(e^{a_1} - 1\right)\right), \tag{3.13}$$

and $g(a_1, K) = 1/e$, where

$$g(u, K) = Ke^{-2u} \frac{\left[(K - 1)(1 - e^{-u}) - 1\right]^2}{(K - 1)^2 (1 - e^{-u})^3}.$$

Indeed, (3.12) comes from the equation given for $t_5 - t_4$ in Proposition 3.6, (3.13) is obtained by substituting (3.12) in the $t_7 - t_6$-equation, and $g(a_1, K) = 1/e$ follows by substituting (3.12) and (3.13) in the $t_3 + 1 - \tau$-equation.

Recall that by definition, $a \in \mathbb{R}^3$ belongs to U_0^3 if and only if

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_1 + a_2 + a_3 < 1,$$
$$x^{\Sigma(a,0)}(s_1) > 1,\ \left|x^{\Sigma(a,0)}(s_2)\right| < 1,\ x^{\Sigma(a,0)}(s_3) < -1,$$
$$-1 < x^{\Sigma(a,0)}(t_1 + 1) = x^{\Sigma(a,0)}(t_2 + 1) < 0 \text{ and } x^{\Sigma(a,0)}(t_3 + 1) > 0.$$

If $a \in \mathbb{R}^3$ with $F(a, 0) = a$, then not only a_2 and a_3 can be expressed as functions of K and a_1, but also $a_1 + a_2 + a_3$, $x^{\Sigma(a,0)}(s_i)$ and $x^{\Sigma(a,0)}(t_i + 1)$ for all $i \in \{1, 2, 3\}$. Computing $e^{a_2 + a_3}$ from (3.12) and (3.13), and substituting for e^{2a_1} from the equation $g(a_1, K) = e^{-1}$, one obtains that

$$a_1 + a_2 + a_3 = 1 + \ln\left(K - \frac{K}{(K - 1)(1 - e^{-a_1})}\right). \tag{3.14}$$

By (3.4) and the definition of I_1, we get $x^{\Sigma(a,0)}(s_1) = e^{-s_1} I_1 = K(1 - e^{-a_1})$. Relations (3.4), (3.12), (3.13) and the definitions of I_2 and I_3 yield

$$x^{\Sigma(a,0)}(s_2) = e^{-s_2} I_2 = K\left[(K - 1)\left(1 - e^{-a_1}\right) - 1\right],$$

$$x^{\Sigma(a,0)}(s_3) = e^{-s_3} I_3 = -K\left(1 - e^{-a_1}\right) = -x^{\Sigma(a,0)}(s_1). \tag{3.15}$$

Also, (3.12), (3.13), $g(a_1, K) = e^{-1}$, Proposition 3.3 and the definitions of $I_{1,*}$ and $I_{3,*}$ give

$$\begin{aligned} x^{\Sigma(a,0)}(t_1+1) = x^{\Sigma(a,0)}(t_2+1) &= \frac{K-1}{K} I_{3,*} = K\left[1-(K-1)\left(1-e^{-a_1}\right)\right] \\ &= -x^{\Sigma(a,0)}(s_2), \\ x^{\Sigma(a,0)}(t_3+1) &= K + \frac{K}{e(K-1)I_{1,*}}\left(x^{\Sigma(a,0)}(t_1+1) - K\right) \\ &= K\left(1-e^{-a_1}\right) = x^{\Sigma(a,0)}(s_1). \end{aligned}$$

As one can check by elementary calculations, these relations imply that $a \in \mathbb{R}^3$ satisfying $F(a,0) = a$ belongs to U_0^3 if and only if

$$a_1 \in J_K = \left(\ln \frac{K-1}{K-2}, \ln \frac{K^2-K}{K^2-2K-1}\right).$$

Hence we get a unique solution $a^* = (a_1^*, a_2^*, a_3^*)$ of $F(a,0) = a$ in U_0^3 if there exist a unique $a_1^* \in J_K$ with $g(a_1^*, K) = e^{-1}$, furthermore a_2^* and a_3^* are defined by (3.12) and (3.13), respectively.

We claim that $g(\cdot, K)$ is strictly increasing on J_K for $K > 3$. Note that

$$\frac{\partial g(u,K)}{\partial u} = g(u,K)\frac{2+e^{-u}+(K-1)(1-e^{-u})e^{-u}-2(K-1)(1-e^{-u})}{[(K-1)(1-e^{-u})-1](1-e^{-u})}.$$

If $u \in J_K$, then $(K-1)(1-e^{-u}) - 1 \in (0, 1/K)$. Hence it suffices to show that for $K > 3$ and $u \in J_K$,

$$2 + e^{-u} + (K-1)\left(1-e^{-u}\right)e^{-u} - 2(K-1)\left(1-e^{-u}\right) > 0,$$

which inequality is equivalent to the second order inequality

$$(2K-4)z^2 - (3K-2)z + (K-1) < 0$$

with $z = e^u$. The solution formula gives that we have to show that for $K > 3$, $J_K \subset (\ln z_1, \ln z_2)$, where

$$z_1 = \frac{3K-2-\sqrt{K^2+12(K-1)}}{4K-8} \text{ and } z_2 = \frac{3K-2+\sqrt{K^2+12(K-1)}}{4K-8}.$$

As $\sqrt{K^2+12(K-1)} > K+2$ for all $K > 2$, we see that

$$\ln z_1 < \ln \frac{3K-2-(K+2)}{4K-8} = \ln \frac{1}{2} < \inf J_K.$$

The same estimate yields that $z_2 > K/(K-2)$, and it is easy to see that

$$\frac{K}{K-2} > \frac{K^2-K}{K^2-2K-1}$$

for $K > 3$. Hence $\ln z_2 > \sup J_K$ and $g'_u(u,K) > 0$ for $K > 3$ and $u \in J_K$.

In addition, $g(u,K) \to 0$ as $u \to \inf J_K + 0$. Also,

$$\lim_{u \to \sup J_K - 0} g(u,K) = w(K) \begin{cases} < \frac{1}{e}, & 3 < K < K^*, \\ = \frac{1}{e}, & K = K^*, \\ > \frac{1}{e} & K > K^*. \end{cases}$$

Therefore the continuity and monotonicity of g implies that for $K > 3$, there exists $a_1^* \in J_K$ with $g(a_1^*, K) = e^{-1}$ if and only if $K > K^*$, and the solution is unique if it exists. $\square$

One may verify using a construction similar to the one given above that for $K = K^*$, $F(\cdot, 0)$ has a fixed point on the boundary of U_0^3.

Proposition 3.9. *For $K > K^*$, $x^{\Sigma(a^*,0)} : \mathbb{R} \to \mathbb{R}$ is an LSOP solution.*

Proof. Consider solution $x = x^{\Sigma(a^*,0)} : \mathbb{R} \to \mathbb{R}$. It follows from the construction introduced above that the minimal period of x is $\tau + 1$ with $\tau > 0$, and x is monotone nonincreasing on $[s_1, s_3]$. Therefore is suffices to prove that $\tau < 1$,

$$2(s_3 - s_1) = \tau + 1 \tag{3.16}$$

and

$$x\left(t + \frac{\tau+1}{2}\right) = -x(t) \tag{3.17}$$

for $t \in [s_1, s_3]$.

By (3.11), (3.15) and $I_3^* = I_3 = x(s_3) e^{s_3}$,

$$\tau = \ln\left(\frac{K}{K-1} - \frac{x(s_3)e^{s_3}}{K}\right) = \ln\left(\frac{K}{K-1} + \left(1 - e^{-a_1^*}\right)e^{s_3}\right)$$

Substituting result (3.14) into the right hand side, we get

$$\tau = \ln\left(K\left(1 - e^{-a_1^*}\right)\right). \tag{3.18}$$

So $\tau < 1$ if and only if $a_1^* < \ln K - \ln(K - e)$. As $a_1^* \in J_K$ (see the proof of Proposition 3.8), this bound holds.

Relations (3.12) and (3.13) imply

$$e^{2(s_3-s_1)} = e^{2\left(a_2^*+a_3^*\right)} = e^{2a_1^*} \frac{(K-1)^2\left(1-e^{-a_1^*}\right)^4}{\left[(K-1)\left(1-e^{-a_1^*}\right)-1\right]^2}$$

Using relation $g(a_1, K) = e^{-1}$ from the proof of Proposition 3.8,

$$2(s_3 - s_1) = \ln\left(Ke\left(1-e^{-a_1^*}\right)\right).$$

This result together with (3.18) gives (3.16).

As $x(s_1) = -x(s_3)$ by (3.15) and x is of type (0) on $[s_1, s_2]$ and on $[s_3, t_1+1]$, the special symmetry follows for $t \in [s_1, s_2]$ if $s_2 - s_1 = t_1 + 1 - s_3$ holds. This equation is the direct consequence of (3.6), (3.12) and (3.14). In particular, $x(s_2) = -x(t_1+1)$. As x is of type $(-K)$ on $[s_2, s_3]$ and of type (K) on $[t_1+1, t_3+1]$, special symmetry holds for $t \in [s_2, s_3]$ if $a_3 = s_3 - s_2 = t_3 - t_1$. This result comes from (3.6), (3.9), the definition of $I_{1,*}$ and (3.13). So (3.17) follows.

The proof is complete. □

Remark 3.10. A numerical study executed with the aid of the CAPD program [1] gives that for $K = 7$,

$$a^* \in [0.2108, 0.2109] \times [0.3003, 0.3004] \times [0.3426, 0.3427].$$

It is shown that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $D_aF(a^*, 0) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ are real with

$$\lambda_1 \in [0.7933, 0.7934],\ \ \lambda_2 \in [3.9187, 3.9188] \text{ and } \lambda_3 \in [6.8362, 6.8363].$$

Now we are capable of verifying the existence of an LSOP solution for Eq. (3.1) for small $\varepsilon > 0$. In the sequel we fix $K = 7$, but the results below can be easily modified for any $K > K^*$. Since we look for an example with large amplitude periodic orbits, a particular K is sufficient.

Proposition 3.11. *Set $K = 7$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0)$, $F(a, \varepsilon) = a$ has a solution $a^*(\varepsilon)$ in $U_\varepsilon^3 = \{a \in \mathbb{R}^3 : (a, \varepsilon) \in U^3\}$, and $x^{\Sigma(a^*(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ is an LSOP solution of Eq. (3.1) with nonlinearity $f^{7,\varepsilon}$. The range $x^{\Sigma(a^*(\varepsilon),\varepsilon)}(\mathbb{R})$ is a subset of $(-7, 7)$ for all $\varepsilon \in [0, \varepsilon_0)$.*

Proof. As U^3 is open in $\mathbb{R}^3 \times [0, 1)$,

$$U = \left\{(a, \varepsilon) : (a, |\varepsilon|) \in U^3\right\}$$

is open in $\mathbb{R}^4$. We extend the definition of F for $\varepsilon < 0$ because we intend to use the implicit function theorem. Let $G : U \to \mathbb{R}^3$ be given by

$$G(a,\varepsilon) = \begin{cases} F(a,\varepsilon) & \text{if } \varepsilon \geq 0, \\ 2F(a,0) - F(a,-\varepsilon) & \text{if } \varepsilon < 0. \end{cases}$$

Then G is C^1-smooth and $G(a^*,0) - a^* = 0$. As 1 is not an eigenvalue of $D_a G(a^*,0)$ by Remark 3.10, the implicit function theorem yields the existence of $\varepsilon_0 > 0$, a convex bounded open neighborhood N of a^* in $\mathbb{R}^3$ and a C^1 function $a^* : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3$ so that $N \times (-\varepsilon_0, \varepsilon_0) \subset U$, $a^*((-\varepsilon_0, \varepsilon_0)) \subset N$, $a^*(0) = a^*$ and for every $(a,\varepsilon) \in N \times (-\varepsilon_0, \varepsilon_0)$, $G(a,\varepsilon) - a = 0$ if and only if $a = a^*(\varepsilon)$. That is $F(a^*(\varepsilon), \varepsilon) = a^*(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_0)$.

Then $x^{\Sigma(a^*(\varepsilon),\varepsilon)} : [-1,\infty) \to \mathbb{R}$ is a periodic solution of Eq. (3.1) with feedback function $f^{7,\varepsilon}$ for all $\varepsilon \in [0,\varepsilon_0)$, and it can be extended to $\mathbb{R}$.

According to Proposition 3.9, $x^{\Sigma(a^*,0)}$ is an LSOP solution. It is also clear from Remark 3.4 and the special symmetry property of $x^{\Sigma(a^*,0)}$ that

$$\max_{t\in\mathbb{R}} x^{\Sigma(a^*,0)}(t) = \Sigma(a^*,0)(s_1) = 7\left(1 - e^{-a_1^*}\right) < 7$$

and therefore $\min_{t\in\mathbb{R}} x^{\Sigma(a^*,0)}(t) > -7$.

For $\varepsilon \in (0,\varepsilon_0)$, Lemma 2.3 (i) and the periodicity of $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ gives that $V\left(x_t^{\Sigma(a^*(\varepsilon),\varepsilon)}\right)$ is the same constant for all $t \in \mathbb{R}$. It follows from the construction that $V(\Sigma(a^*(\varepsilon),\varepsilon)) = 2$. Thus $V\left(x_t^{\Sigma(a^*(\varepsilon),\varepsilon)}\right) = 2$ for all $t \in \mathbb{R}$. It remains to confirm that

$$(\xi_{-1},\xi_1) \subset x^{\Sigma(a^*(\varepsilon),\varepsilon)}(\mathbb{R}) \subset (-7,7)$$

for all $\varepsilon \in (0,\varepsilon_0)$. Proposition 2.6 ensures that

$$x^{\Sigma(a^*(\varepsilon),\varepsilon)}(\mathbb{R}) \subset (-7,7) = (\xi_{-2},\xi_2)$$

holds for all $\varepsilon \in (0,\varepsilon_0)$, that is the segments of $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ belong to $C_{-2,2}$ for all $\varepsilon \in (0,\varepsilon_0)$. Then $x^{\Sigma(a^*(\varepsilon),\varepsilon)}(\mathbb{R}) \supset (\xi_{-1},\xi_1)$ by Proposition 2.5. Hence $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ is an LSOP solution with range in $(-7,7)$ for all $\varepsilon \in (0,\varepsilon_0)$. □

Remark 3.12. $D_a F(a^*(\varepsilon),\varepsilon)$ has at most 3 distinct (possibly complex) eigenvalues, and as F is smooth (see Proposition 3.7), they are close to the eigenvalues of $D_a F(a^*,0)$ in $\mathbb{C}$ for $\varepsilon > 0$ small. Because of Remark 3.10, we may choose $\varepsilon_0 > 0$ sufficiently small such that for $\varepsilon \in [0,\varepsilon_0)$, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $D_a F(a^*(\varepsilon),\varepsilon)$ are real, simple and satisfy

$$0 < \lambda_1 < 0.9, \quad 3 < \lambda_2 < 5 < \lambda_3.$$

Consider the case $\varepsilon = 0$. As equation $\dot{x}(t) = -x(t)$ admits no nontrivial periodic solution, any periodic solution x of Eq. (3.1) with initial function in $\Sigma\left(U_0^1\right)$ necessarily satisfies $x\left(s_1\right) > 1$ or $x\left(s_3\right) < -1$. However, condition $x\left(s_2\right) < 1$ is not self-evident. This recognition leads to an alternative construction yielding a second periodic solution of Eq. (3.1) for $K > K^*$ and $\varepsilon = 0$ and a second LSOP solution of Eq. (3.1) for $K = 7$ and $\varepsilon > 0$ small. Next we introduce this construction but omit the detailed calculations as they are analogous to the previous ones.

3.2. **Another LSOP solution for nonlinearity $f^{K,\varepsilon}$.** For $K > 3$, define

$$\widetilde{U}^1 = \left\{(a,\varepsilon) \in (0,1)^3 \times [0,1) : \, a_1 + a_2 + a_3 + 2\tilde{T}\left(\varepsilon\right) + \hat{T}\left(\varepsilon\right) < 1\right\}$$

and

$$\widetilde{U}_\varepsilon^1 = \left\{a \in \mathbb{R}^3 : (a,\varepsilon) \in \widetilde{U}^1\right\}, \quad \varepsilon \in [0,1)\,.$$

Note that $\widetilde{U}_0^1 = U_0^1$. For given $(a,\varepsilon) \in \widetilde{U}^1$, set

$$\begin{aligned}
s_0 &= -1,\\
s_1 &= s_0 + a_1 = -1 + a_1,\\
s_1^* &= s_1 + \tilde{T}\left(\varepsilon\right) = -1 + a_1 + \tilde{T}\left(\varepsilon\right),\\
s_2 &= s_1^* + a_2 = -1 + a_1 + \tilde{T}\left(\varepsilon\right) + a_2,\\
s_2^* &= s_2 + \hat{T}\left(\varepsilon\right) = -1 + a_1 + \tilde{T}\left(\varepsilon\right) + a_2 + \hat{T}\left(\varepsilon\right),\\
s_3 &= s_2^* + a_3 = -1 + a_1 + \tilde{T}\left(\varepsilon\right) + a_2 + \hat{T}\left(\varepsilon\right) + a_3,\\
s_3^* &= s_3 + \tilde{T}\left(\varepsilon\right) = -1 + a_1 + \tilde{T}\left(\varepsilon\right) + a_2 + \hat{T}\left(\varepsilon\right) + a_3 + \tilde{T}\left(\varepsilon\right).
\end{aligned}$$

Similarly, define $\widetilde{h} = \widetilde{h}\left(a,\varepsilon\right) : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{h}\left(t\right) = \begin{cases}
K & \text{if } t < s_1,\\
f^{K,\varepsilon}\left(-K + \left(K+1+\varepsilon\right)e^{-(t-s_1)}\right) & \text{if } s_1 \le t < s_1^*,\\
0 & \text{if } s_1^* \le t < s_2,\\
f^{K,\varepsilon}\left(-K + \left(K-1\right)e^{-(t-s_2)}\right) & \text{if } s_2 \le t < s_2^*,\\
-K & \text{if } s_2^* \le t < s_3,\\
f^{K,\varepsilon}\left(K - \left(K+1+\varepsilon\right)e^{-(t-s_3)}\right) & \text{if } s_3 \le t < s_3^*,\\
0 & \text{if } s_3^* \le t.
\end{cases}$$

and the continuous map $\widetilde{\Sigma} : \widetilde{U}^1 \to C$ by

$$\widetilde{\Sigma}\left(a,\varepsilon\right)\left(t\right) = e^{-t}\int_{-1}^{t} e^s \widetilde{h}\left(a,\varepsilon\right)\left(s\right) ds \qquad \left(-1 \le t \le 0\right).$$

Note that for $a \in \widetilde{U}_0^1 = U_0^1$, $\widetilde{\Sigma}(a,0) = \Sigma(a,0)$.

Proposition 3.13. *For each fixed* $\varepsilon \in (0,1)$*, the map* $\widetilde{U}_\varepsilon^1 \ni a \mapsto \widetilde{\Sigma}(a,\varepsilon) \in C$ *is* C^1*-smooth with*

$$D_a\widetilde{\Sigma}(a,\varepsilon)(b) = b_1\widetilde{\psi}_1 + b_2\widetilde{\psi}_2 + b_3\widetilde{\psi}_3$$

for all $a \in \widetilde{U}_\varepsilon^1$ *and* $b = (b_1,b_2,b_3) \in \mathbb{R}^3$*, where*

$$\widetilde{\psi}_i : [-1,0] \ni t \mapsto e^{-t}\int_{-1}^{t} e^s \frac{\partial}{\partial a_i}\widetilde{h}(a,\varepsilon)(s)\,ds \in \mathbb{R}, \quad i \in \{1,2,3\},$$

are linearly independent elements of C*.*

Now let

$$\widetilde{U}^2 = \Big\{(a,\varepsilon) \in \widetilde{U}^1 : \widetilde{\Sigma}(a,\varepsilon)(s) > 1+\varepsilon \text{ for } s \in [s_1,s_1^*] \cup [s_2,s_2^*], \\ \widetilde{\Sigma}(a,\varepsilon)(s) < -1-\varepsilon \text{ for } s \in [s_3,s_3^*]\Big\}.$$

If $(a,\varepsilon) \in \widetilde{U}^2$ and $x = x^{\widetilde{\Sigma}(a,\varepsilon)} : [-1,\infty) \to \mathbb{R}$ is the solution of Eq. (3.1) with initial function $\widetilde{\Sigma}(a,\varepsilon)$, then there exist $t_1, t_2, ..., t_6$ in $[-1,0]$ such that

$$-1 < t_1 \leq t_2 < s_1 \leq s_1^* < s_2 \leq s_2^* < t_3 \leq t_4 < t_5 \leq t_6 < s_3 \leq s_3^*$$

and

$$x(t_1) = 1,\ x(t_2) = 1+\varepsilon,\ x(t_3) = 1+\varepsilon,\ x(t_4) = 1,\ x(t_5) = -1,\ x(t_6) = -1-\varepsilon$$

(see Fig. 7). A second subset of $\widetilde{U}^1$ is

$$\widetilde{U}^3 = \Big\{(a,\varepsilon) \in \widetilde{U}^2 : \quad x^{\widetilde{\Sigma}(a,\varepsilon)}(t_2+1) < -1-\varepsilon,\ x^{\widetilde{\Sigma}(a,\varepsilon)}(t_3+1) > 0\Big\}.$$

One may show that $\widetilde{U}^3$ is open in $(0,1)^3 \times [0,1)$. Fig. 7 shows a typical element of $\widetilde{\Sigma}\left(\widetilde{U}^3\right)$.

The following remark resembles Remark 3.4 and we are going to refer to it throughout the paper.

Remark 3.14. Observe that any $\varphi \in \widetilde{\Sigma}\left(\widetilde{U}^3\right)$ can be characterized as follows: there exist $\varepsilon \in [0,1)$ and

$$-1 < s_1 \leq s_1^* < s_2 \leq s_2^* < s_3 \leq s_3^* < 0$$

with

$$s_1^* - s_1 = \tilde{T}(\varepsilon),\ s_2^* - s_2 = \hat{T}(\varepsilon),\ s_3^* - s_3 = \tilde{T}(\varepsilon)$$

so that $\varphi \in C$ and

(i) $\varphi(-1) = 0$,
(ii) φ is of type (K) on $[-1, s_1]$,
(iii) φ is of type $(-K, 1+\varepsilon)$ on $[s_1, s_1^*]$,
(iv) φ is of type (0) on $[s_1^*, s_2]$,
(v) φ is of type $(-K, -1)$ on $[s_2, s_2^*]$,
(vi) φ is of type $(-K)$ on $[s_2^*, s_3]$,
(vii) φ is of type $(K, -1-\varepsilon)$ on $[s_3, s_3^*]$,
(viii) φ is of type (0) on $[s_3^*, 0]$,
(ix) $\varphi(s) > 1+\varepsilon$ for $s \in [s_1, s_1^*] \cup [s_2, s_2^*]$,
(x) $\varphi(s) < -1-\varepsilon$ for $s \in [s_3, s_3^*]$,
(xi) if $-1 \le t_2 < s_1$ with $\varphi(t_2) = 1+\varepsilon$, then $x^{\varphi}(t_2+1) < 0$,
(xii) if $s_1^* < t_3 < s_2$ with $\varphi(t_3) = 1+\varepsilon$, then $x^{\varphi}(t_3+1) > 0$.
Note that (i)-(viii) characterize $\varphi \in \widetilde{\Sigma}\left(\widetilde{U}^1\right)$ and (i)-(x) characterize $\varphi \in \widetilde{\Sigma}\left(\widetilde{U}^2\right)$.

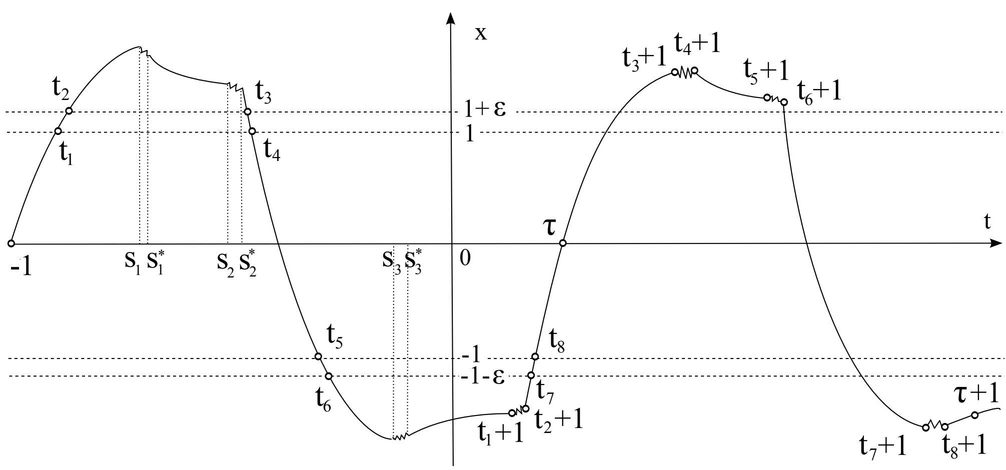

FIGURE 7. Solution $x^{\widetilde{\Sigma}(a,\varepsilon)}$ of Eq. (3.1)

For $(a,\varepsilon) \in \widetilde{U}^3$, let τ be the (unique) zero of $x = x^{\widetilde{\Sigma}(a,\varepsilon)}$ on $[t_2+1, t_3+1]$. If $(a,\varepsilon) \in \widetilde{U}^3$ and $t_1, t_2, ..., t_8, \tau$ are defined as in this subsection, then $x_{\tau+1} \in \widetilde{\Sigma}\left(\widetilde{U}^1\right)$ and

$$x_{\tau+1} = \widetilde{\Sigma}\left(t_3+1-\tau, t_5-t_4, t_7-t_6, \varepsilon\right).$$

As in the previous subsection, τ and t_i, $i \in \{1,..,6\}$, are C^1-smooth functions of (a,ε). Therefore we may introduce the C^1-smooth map $\widetilde{F} : \widetilde{U}^3 \to \mathbb{R}^3$, $\tilde{F}(a,\varepsilon) = (t_3+1-\tau, t_5-t_4, t_7-t_6)$. In case $\widetilde{F}(a,\varepsilon) = a$ for $(a,\varepsilon) \in \widetilde{U}^3$, then $x^{\widetilde{\Sigma}(a,\varepsilon)}$ is a periodic solution of Eq. (3.1).

Introduce the notation

$$\widetilde{U}_{\varepsilon}^{3}=\left\{a\in\mathbb{R}^{3}:(a,\varepsilon)\in\widetilde{U}^{3}\right\},\quad \varepsilon\in[0,1),$$

and recall the definition of K^* from the previous subsection. We obtain the following results analogously to Proposition 3.8 and Proposition 3.11.

Proposition 3.15. *For $K>K^*$, there exists a unique $\widetilde{a}\in\widetilde{U}_0^3$ with $\widetilde{F}(\widetilde{a},0)=\widetilde{a}$. For $K\in(3,K^*]$, $\widetilde{F}(a,0)=a$ has no solution in $\widetilde{U}_0^3$.*

It can be shown that for $K=K^*$, $\widetilde{F}(\cdot,0)$ has a fixed point on the boundary of $\widetilde{U}_0^3$ and it equals the fixed point of $F(\cdot,0)$.

Proposition 3.16. *For $K>K^*$, $x^{\widetilde{\Sigma}(\widetilde{a},0)}:\mathbb{R}\to\mathbb{R}$ is an LSOP solution.*

Remark 3.17. For $K=7$, a numerical study executed with the aid of the CAPD program [1] gives that

$$\widetilde{a}\in[0.2202,0.2203]\times[0.2876,0.2877]\times[0.3585,0.3586].$$

In addition, it is shown that the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ of $D_a\widetilde{F}(\widetilde{a},0)\in\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$ are real with

$$\lambda_1=0,\ \lambda_2\in[-0.2415,0.2347]\ \text{ and }\lambda_3\in[2.3226,2.3227].$$

Proposition 3.18. *For $K=7$, there exists $\widetilde{\varepsilon}_0>0$ such that for all $\varepsilon\in[0,\widetilde{\varepsilon}_0)$, $\widetilde{F}(a,\varepsilon)=a$ has a solution $\widetilde{a}(\varepsilon)$ in $\widetilde{U}_{\varepsilon}^3$, and $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}:\mathbb{R}\to\mathbb{R}$ is an LSOP solution. The range $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}(\mathbb{R})$ is a proper subset of $(-7,7)$.*

Remark 3.19. It follows from the smoothness of $\widetilde{F}$ and Remark 3.17, that one may set $\widetilde{\varepsilon}_0>0$ so small that for $\varepsilon\in[0,\widetilde{\varepsilon}_0)$, the eigenvalues λ_1,λ_2 and λ_3 of $D_a\widetilde{F}(\widetilde{a}(\varepsilon),\varepsilon)$ satisfy

$$0\le|\lambda_1|\le|\lambda_2|<.5,\quad 2<\lambda_3.$$

Note that λ_3 is necessarily real. Either both λ_1 and λ_2 are real, or $\lambda_2=\overline{\lambda_1}$.

Remark 3.20. It is clear from Remarks 3.4 and 3.14 that for $K=7$,

$$\max_{t\in\mathbb{R}}x^{\Sigma(a^*,0)}(t)=\Sigma(a^*,0)(s_1)=7\left(1-e^{-a_1^*}\right)$$

and

$$\max_{t\in\mathbb{R}}x^{\widetilde{\Sigma}(\widetilde{a},0)}(t)=\widetilde{\Sigma}(\widetilde{a},0)(s_1)=7\left(1-e^{-\widetilde{a}_1}\right).$$

As $a_1^* < \widetilde{a}_1$ by Remarks 3.10 and 3.17, we see that

$$\max_{t\in\mathbb{R}} x^{\Sigma(a^*,0)}(t) < \max_{t\in\mathbb{R}} x^{\widetilde{\Sigma}(\widetilde{a},0)}(t).$$

Both periodic solutions are of special symmetry, hence

$$\min_{t\in\mathbb{R}} x^{\Sigma(a^*,0)}(t) = \Sigma(a^*,0)(s_3) > \widetilde{\Sigma}(\widetilde{a},0)(s_3) = \min_{t\in\mathbb{R}} x^{\widetilde{\Sigma}(\widetilde{a},0)}(t).$$

As Σ and $\widetilde{\Sigma}$ are continuous functions of (a,ε), furthermore a^* and $\widetilde{a}$ are continuous functions of ε, we may suppose that the same inequalities,

$$\max_{t\in\mathbb{R}} x^{\Sigma(a^*(\varepsilon),\varepsilon)}(t) = \max_{t\in[-1,0]} \Sigma(a^*(\varepsilon),\varepsilon)(t) < \max_{t\in[-1,0]} \widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)(t) = \max_{t\in\mathbb{R}} x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}(t)$$

and

$$\min_{t\in\mathbb{R}} x^{\Sigma(a^*(\varepsilon),\varepsilon)}(t) = \min_{t\in[-1,0]} \Sigma(a^*(\varepsilon),\varepsilon)(t) > \min_{t\in[-1,0]} \widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)(t) = \min_{t\in\mathbb{R}} x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}(t)$$

hold for all $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$.

Remark on the choice of K. We can summarize our results regarding case $\varepsilon = 0$ as follows. For $K \in (3, K^*)$, Eq. (3.1) admits no periodic solution with initial function in $\Sigma(U_0^3, 0) \cup \widetilde{\Sigma}\left(\widetilde{U}_0^3, 0\right)$. For $K > K^*$, Eq. (3.1) has a unique periodic solution with initial segment in $\Sigma(U_0^3, 0)$ and a unique periodic solution with initial segment in $\widetilde{\Sigma}\left(\widetilde{U}_0^3, 0\right)$. It can be shown that for $K = K^*$, there is a single periodic solution with initial function in $\mathrm{bd}\Sigma(U_0^3, 0) \cap \mathrm{bd}\widetilde{\Sigma}\left(\widetilde{U}_0^3, 0\right)$.

Fig. 8 shows the graphs of the first components of the fixed points of $F(\cdot, 0)$ and $\widetilde{F}(\cdot, 0)$ for $K \geq K^*$ (as functions of K).

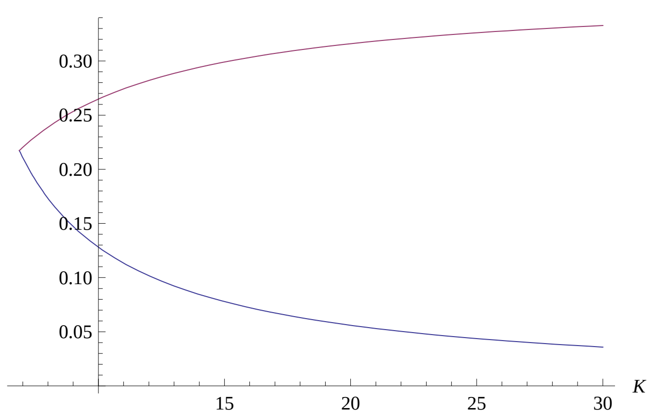

FIGURE 8. The plot of $a_1^* = a_1^*(K)$ and $\widetilde{a}_1 = \widetilde{a}_1(K)$ for $K \geq K^*$

This suggests that for each fixed small $\varepsilon > 0$ there exists $K^*(\varepsilon)$ so that Eq. (3.1) undergoes a saddle-node-like bifurcation of periodic orbits at $K = K^*(\varepsilon)$.

To give a more detailed picture of the case $\varepsilon = 0$, we are going to show the following results in Section 6. For $K > K^*$ and $\varepsilon = 0$, $x^{\Sigma(a^*,0)} : \mathbb{R} \to \mathbb{R}$ and $x^{\widetilde{\Sigma}(\widetilde{a},0)} : \mathbb{R} \to \mathbb{R}$ are the only normalized LSOP solutions of Eq. (3.1) (see Proposition 6.4). For $0 < K < K^*$ and $\varepsilon = 0$, Eq. (3.1) has no such nontrivial periodic solutions (see Corollary 6.2 and Proposition 6.4).

4. The Existence of LSOP Solutions for a Monotone Nonlinearity

Theorem 1.1 states that one may give a strictly increasing feedback function f so that (1.1) has exactly two LSOP solutions. In this section we discuss the existence of these LSOP solutions.

Let $K = 7$ and $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ be fixed, where ε_0 and $\widetilde{\varepsilon}_0$ are given by Propositions 3.11 and 3.18, respectively. Proposition 3.11 implies that Eq. (3.1) has an LSOP solution with initial function $\Sigma(a^*(\varepsilon), \varepsilon)$ and with range in (ξ_{-2}, ξ_2).

Observe that $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ is a normalized LSOP solution of (3.1) with

$$\Sigma(a^*(\varepsilon), \varepsilon) \in H = \{\varphi \in C : \varphi(-1) = 0\}, \ \frac{\mathrm{d}}{\mathrm{d}t}\Sigma(a^*(\varepsilon), \varepsilon) \notin H.$$

Then a Poincaré return map can be defined on $\{\Sigma(a^*(\varepsilon), \varepsilon)\} + N$, where N is a convex bounded open neighborhood of 0 in H, see Section 2. As P is C^1-smooth and has fixed point $\Sigma(a^*(\varepsilon), \varepsilon)$, there exists a convex open neighborhood $\hat{N} \subset N$ of 0 in H so that $P^2 = P \circ P$ is defined on $\{\Sigma(a^*(\varepsilon), \varepsilon)\} + \hat{N}$. We have the following observation regarding the range of P^2.

Proposition 4.1. *There exists an open neighborhood $V \subseteq \hat{N}$ of 0 in H so that*

$$\text{if } \varphi \in \{\Sigma(a^*(\varepsilon), \varepsilon)\} + V, \text{ then } P^2(\varphi) \in \Sigma\left(U_\varepsilon^3 \times \{\varepsilon\}\right).$$

Proof. If $\varphi \in \{\Sigma(a^*(\varepsilon), \varepsilon)\} + V$, with an appropriate open ball V centered at 0 in H, then x_1^{φ} and $x_1^{\Sigma(a^*(\varepsilon),\varepsilon)}$ are close in C^1-norm, and there exist

$$-1 < \bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \bar{t}_4 < \bar{t}_5 < \bar{t}_6 < \bar{t}_7 < \bar{t}_8 < 0 < \tau_\varphi$$

such that

$$\varphi(\bar{t}_1) = 1, \ \varphi(\bar{t}_2) = 1 + \varepsilon, \ \varphi(\bar{t}_3) = 1 + \varepsilon, \ \varphi(\bar{t}_4) = 1,$$
$$\varphi(\bar{t}_5) = -1, \ \varphi(\bar{t}_6) = -1 - \varepsilon, \ \varphi(\bar{t}_7) = -1 - \varepsilon, \ \varphi(\bar{t}_8) = -1, \ x^{\varphi}(\tau_\varphi) = 0,$$

$$
\begin{aligned}
\varphi(t) &\in (-1,1) && \text{for } t \in \left[-1, \bar{t}_1\right), \\
\varphi(t) &> 1+\varepsilon && \text{for } t \in \left(\bar{t}_2, \bar{t}_3\right), \\
\varphi(t) &\in (-1,1) && \text{for } t \in \left(\bar{t}_4, \bar{t}_5\right), \\
\varphi(t) &< -1-\varepsilon && \text{for } t \in \left(\bar{t}_6, \bar{t}_7\right), \\
x^{\varphi}(t) &\in (-1,1) && \text{for } t \in \left(\bar{t}_8, \tau_\varphi\right],
\end{aligned}
$$

and the smallest positive zero τ_φ of x^φ is simple and belongs to $\left(\bar{t}_2+1, \bar{t}_3+1\right)$. In consequence, $P(\varphi) = x^{\varphi}_{\tau_\varphi+1}$, and we have

$P(\varphi)(-1) = 0$,
$P(\varphi)$ is of type (7) on $\left[-1, \bar{t}_3-\tau_\varphi\right]$,
$P(\varphi)$ is of type (0) on $\left[\bar{t}_4-\tau_\varphi, \bar{t}_5-\tau_\varphi\right]$,
$P(\varphi)$ is of type (-7) on $\left[\bar{t}_6-\tau_\varphi, \bar{t}_7-\tau_\varphi\right]$,
$P(\varphi)$ is of type (0) on $\left[\bar{t}_8-\tau_\varphi, 0\right]$.

If the radius of V is small enough, then also
$P(\varphi)(t) > 1+\varepsilon$ for $t \in \left[\bar{t}_3-\tau_\varphi, \bar{t}_4-\tau_\varphi\right]$,
$|P(\varphi)(t)| < 1$ for $t \in \left[\bar{t}_5-\tau_\varphi, \bar{t}_6-\tau_\varphi\right]$
and $P(\varphi)(t) < -1-\varepsilon$ for $t \in \left[\bar{t}_7-\tau_\varphi, \bar{t}_8-\tau_\varphi\right]$.

In this case it also follows that whenever $P(\varphi)$ maps the disjoint subintervals J_1, J_2, J_3, J_4 of $[-1,0]$ onto the intervals $[1,1+\varepsilon]$, $[1,1+\varepsilon]$, $[-1-\varepsilon,-1]$, $[-1-\varepsilon,-1]$, respectively, then $P(\varphi)$ is of type (7), (0), (-7), (0) on J_1, J_2, J_3, J_4, respectively, and therefore $x^{P(\varphi)}$ is of type $(7,1)$, $(0,1+\varepsilon)$, $(-7,-1)$, $(0,-1-\varepsilon)$ on J_1+1, J_2+1, J_3+1, J_4+1, respectively. Using an argument similar to the one given above, now it is easy to see that if we take neighborhood V small enough, then $P^2(\varphi)$ satisfies conditions (i)-(viii) of Remark 3.4 with some

$$
-1 < \bar{s}_1 < \bar{s}_1^* < \bar{s}_2 < \bar{s}_2^* < \bar{s}_3 < \bar{s}_3^* < 0,
$$

where

$$
\bar{s}_1^* - \bar{s}_1 = T(\varepsilon), \ \bar{s}_2^* - \bar{s}_2 = \hat{T}(\varepsilon), \ \bar{s}_3^* - \bar{s}_3 = T(\varepsilon).
$$

Using the smooth dependence of solutions on initial data and decreasing the radius of V further, we can achieve that $P^2(\varphi)$ satisfies conditions (ix)-(xiv) of Remark 3.4 and thus $P^2(\varphi) \in \Sigma(U_\varepsilon^3 \times \{\varepsilon\})$. □

Note that for any small neighborhood V of 0 in H, there is $\varphi \in \{\Sigma(a^*(\varepsilon), \varepsilon)\} + V$ so that $P(\varphi)$ does not satisfy conditions (iii), (v) and (vii) of Remark 3.4. So we cannot state that $P(\varphi) \in \Sigma(U_\varepsilon^3 \times \{\varepsilon\})$.

Proposition 3.18 yields that Eq. (3.1) has another LSOP solution with initial segment $\widetilde{\Sigma}(\widetilde{a}(\varepsilon), \varepsilon)$ and with range in (ξ_{-2}, ξ_2). Then one may define a Poincaré return map

P in a neighborhood of $\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)$ in H in an analogous fashion. The analogue of Proposition 4.1 holds.

Proposition 4.2. *There is an open neighborhood $\widetilde{V}$ of 0 in H such that*

$$\text{if } \varphi \in \left\{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)\right\} + \widetilde{V}, \text{ then } P^2(\varphi) \in \widetilde{\Sigma}\left(\widetilde{U}^3_\varepsilon \times \{\varepsilon\}\right).$$

We omit the proof.

The hyperbolicity of the LSOP solutions is confirmed with the aid of the next proposition.

Proposition 4.3. *Suppose that X is a real Banach space, $\mathcal{V}_0, \mathcal{V}_1$ and $\mathcal{U}_0, \mathcal{U}_1$ are open subsets of X and $\mathbb{R}^m$, respectively, $\mathcal{V}_1 \subset \mathcal{V}_0$, $\mathcal{U}_1 \subset \mathcal{U}_0$, $x_0 \in \mathcal{V}_1$, $u_0 \in \mathcal{U}_1$, the maps*

$$Q : \mathcal{U}_0 \to \mathbb{R}^m,\ R : \mathcal{U}_0 \to X,\ S : \mathcal{V}_0 \to X$$

are C^1-smooth, $Q(u_0) = u_0$, $R(u_0) = x_0$, $S(x_0) = x_0$, $Q(\mathcal{U}_1) \subset \mathcal{U}_0$, $R(\mathcal{U}_1) \subset \mathcal{V}_0$, $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$, moreover, $DR(u_0) \in \mathcal{L}(\mathbb{R}^m, X)$ is injective and $S(R(u)) = R(Q(u))$ for all $u \in \mathcal{U}_1$. Then

$$\sigma(DS(x_0)) = \{0\} \cup \sigma(DQ(u_0)),$$

and for each $\lambda \in \sigma(DS(x_0))\backslash\{0\}$, the corresponding generalized eigenspaces of $DS(x_0)$ and $DQ(u_0)$ have the same dimension.

Proof. By introducing the maps

$$u \mapsto Q(u+u_0) - Q(u_0),\ u \mapsto R(u+u_0) - R(u_0),\ x \mapsto S(x+x_0) - S(x_0),$$

we may assume that $x_0 = 0$ and $u_0 = 0$.

By the injectivity of $DR(0)$, the set $Y = \{DR(0)u :\ u \in \mathbb{R}^m\}$ is an m-dimensional subspace of X and

$$A : \mathbb{R}^m \ni u \mapsto DR(0)u \in Y$$

is a linear isomorphism. Let A^{-1} denote the inverse of A. Since Y is finite dimensional, there is a closed complementary subspace Z of Y in X, i. e., $X = Y \oplus Z$. The set $Y_0 = A(\mathcal{U}_0)$ is an open neighborhood of $0 \in Y$. Define the map

$$T : Y_0 + Z \ni y + z \mapsto R\left(A^{-1}(y)\right) + z \in X.$$

Clearly T is C^1-smooth, $T(0) = 0$, $DT(0) = id_X$ and $T(Y_0) = R(\mathcal{U}_0)$. The inverse mapping theorem shows that T is a local C^1-isomorphism at $0 \in X$.

If x is in a small neighborhood of $0 \in X$ and $x \in R(\mathcal{U}_1)$, then there exist $y \in Y_0$ and $u \in \mathcal{U}_1$ so that $x = R(u)$, $y = T^{-1}(x)$, $u = A^{-1}y$. Then by applying $S(R(u)) =$

$R(Q(u))$, we find that

$$S(x) = S(R(u)) = R(Q(u)) = R\left(A^{-1}(A(Q(u)))\right) = T(A(Q(u))) \tag{4.1}$$
$$= T\left(A\left(Q\left(A^{-1}y\right)\right)\right) = T \circ A \circ Q \circ A^{-1} \circ T^{-1}(x).$$

In a sufficiently small open neighborhood of $0 \in X$ define the C^1-smooth map s into X by

$$s(x) = T^{-1}(S(T(x))).$$

If x is in the domain of s and $T(x) \in \mathcal{V}_1$, then by the assumption $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$ there exists $u \in \mathcal{U}_1$ so that

$$S(T(x)) = R(u) = R\left(A^{-1}(A(u))\right) = T(A(u)).$$

Hence for such an x we obtain that $s(x) = Au \in Y$. Therefore s maps a small neighborhood of $0 \in X$ into Y. Consequently, $Ds(0)(y+z) = By + Cz$ for all $y \in Y$ and $z \in Z$, where $B \in \mathcal{L}(Y,Y)$ is the derivative of s restricted to a neighborhood of $0 \in Y$ and $C \in \mathcal{L}(Z,Y)$ is the derivative of s restricted to a neighborhood of $0 \in Z$.

If $y \in Y$ is in a sufficiently small neighborhood of $0 \in Y$, then there is $u \in \mathcal{U}_1$ with $y = Au$,

$$T(y) = T(A(u)) = R\left(A^{-1}(A(u))\right) = R(u) \in R(\mathcal{U}_1),$$

and consequently, by applying (4.1),

$$s(y) = T^{-1} \circ S \circ T(y) = T^{-1} \circ T \circ A \circ Q \circ A^{-1} \circ T^{-1} \circ T(y) = A \circ Q \circ A^{-1}(y).$$

Therefore $B = A \circ DQ(0) \circ A^{-1}$. From $DT(0) = DT^{-1}(0) = id_X$ one gets $DS(0) = Ds(0)$. Thus

$$DS(0)(y+z) = \left(A \circ DQ(0) \circ A^{-1}\right) y + Cz$$

for all $y \in Y$, $z \in Z$, with $\operatorname{range}(C) \subset Y$, and the statements of the proposition follow in a straightforward way. □

Proposition 4.4. *The orbits defined by LSOP solutions $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ and $x^{\widetilde{\Sigma}(\tilde{a}(\varepsilon),\varepsilon)}$ are hyperbolic with 2 and 1 Floquet multipliers outside the unit circle, respectively.*

Proof. First we prove that $DP^2(\Sigma(a^*(\varepsilon),\varepsilon))$ has real Floquet multipliers μ_1, μ_2, μ_3 of multiplicity 1 with

$$0 < \mu_1 < 0.81, \quad 9 < \mu_2 < 25 < \mu_3.$$

Set $X = H$, $m = 3$, $x_0 = \Sigma(a^*(\varepsilon),\varepsilon)$ and u_0 to be the fixed point $a^*(\varepsilon)$ of $F(\cdot,\varepsilon)$ in U_ε^3 given by Proposition 3.11. Choose $\mathcal{V}_0 = \{x_0\} + V$, where the open set V is given by Proposition 4.1. Set $\mathcal{U}_0$ to be the open set on which $F^2(\cdot,\varepsilon)$ is defined, that is

$$\mathcal{U}_0 = \left\{a \in U_\varepsilon^3 : \ F(a,\varepsilon) \in U_\varepsilon^3\right\}.$$

Let

$$\mathcal{U}_1 = \left\{a \in \mathcal{U}_0 : \ F^2(a,\varepsilon) \in \mathcal{U}_0 \text{ and } \Sigma(a,\varepsilon) \in \mathcal{V}_0\right\}.$$

Clearly $\mathcal{U}_1 \subset \mathcal{U}_0$ is open and $u_0 \in \mathcal{U}_1$. Let $\mathcal{V}_1 \subset \mathcal{V}_0$ be an open ball with $x_0 \in \mathcal{V}_1$ and $P^2(\mathcal{V}_1) \subset \Sigma(\mathcal{U}_1 \times \{\varepsilon\})$. The latter set exists because $x_0 \in \Sigma(\mathcal{U}_1 \times \{\varepsilon\})$, P^2 is continuous and maps $\mathcal{V}_0$ into $\Sigma(U_\varepsilon^3 \times \{\varepsilon\})$ by Proposition 4.1.

Define

$$Q = F^2 : \mathcal{U}_0 \to \mathbb{R}^3, \quad R = \Sigma(\cdot,\varepsilon) : \mathcal{U}_0 \to H, \quad S = P^2 : \mathcal{V}_0 \to H.$$

Proposition 3.7 yields that Q is C^1-smooth, Proposition 3.2 gives that R is C^1-smooth and $DR(u_0)$ is injective. The map S is also smooth [19]. Clearly $Q(u_0) = u_0$, $R(u_0) = x_0$ and $S(x_0) = x_0$, moreover $\mathcal{U}_1$ and $\mathcal{V}_1$ are chosen so that $Q(\mathcal{U}_1) \subset \mathcal{U}_0$, $R(\mathcal{U}_1) \subset \mathcal{V}_0$ and $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$ hold. It is easy to see that $S(R(u)) = R(Q(u))$ for all $u \in \mathcal{U}_1$.

Remark 3.12 implies that the eigenvalues of $DQ(u_0)$ are μ_i, $i \in \{1,2,3\}$, with $0 < \mu_1 < 0.81$ and $9 < \mu_2 < 25 < \mu_3$. It follows from Proposition 4.3 that the eigenvalues of $DP^2(x_0)$ are $0, \mu_1, \mu_2, \mu_3$ with the above bounds, and μ_i, $i \in \{1,2,3\}$, are simple eigenvalues.

If μ is an eigenvalue of $DP(x_0)$, then μ^2 is an eigenvalue of $DP^2(x_0) = DP(x_0) \circ DP(x_0)$, and the generalized eigenspace of $DP(x_0)$ associated to μ is clearly a subset of the generalized eigenspace of $DP^2(x_0)$ associated to μ^2. Consequently, $DP(x_0)$ has two simple real eigenvalues outside the unit circle, and it has no eigenvalue with absolute value 1.

The statement for $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}$ can be verified in a similar way. □

Choose $D = \mathbb{R}$ and the consider the Banach space $C_b^1(D,\mathbb{R}) = C_b^1(\mathbb{R},\mathbb{R})$. Clearly $f^{7,\varepsilon} \in C_b^1(\mathbb{R},\mathbb{R})$ for all $\varepsilon \in [0,1)$.

Proposition 4.5. *Set $\mu = 1$, $K = 7$. Then for each $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$, where ε_0 and $\widetilde{\varepsilon}_0$ are given by Propositions 3.11 and 3.18, respectively, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ so that if $f \in C_b^1(\mathbb{R},\mathbb{R})$ satisfies (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_0$, then Eq. (1.1) admits two normalized LSOP solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ with $p(\mathbb{R}) \subsetneq q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

are hyperbolic and have 2 and 1 Floquet multipliers outside the unit circle, respectively.

Proof. Consider nonlinearities $f \in C_b^1(\mathbb{R},\mathbb{R})$ satisfying hypothesis (H1). Then Proposition 2.1 and Proposition 4.4 imply that there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_0$, then Eq. (1.1) has two periodic solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ with

$$p_0 \to \Sigma(a^*(\varepsilon),\varepsilon) \text{ and } q_0 \to \widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon) \text{ in } C \text{ as } \left\|f - f^{K,\varepsilon}\right\|_{C_b^1} \to 0.$$

As the initial segments p_0 and q_0 are arbitrarily close to $\Sigma(a^*(\varepsilon),\varepsilon)$ and $\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)$, respectively, and the periodic solutions are of monotone type, we get $V(p_0) = V(q_0) = 2$ if δ_0 is small enough. In this case the periodicity of p and q and the monotonicity of V gives that $V(p_t) = V(q_t) = 2$ for all $t \in \mathbb{R}$. In addition, it is easy to see that one may choose δ_0 so small that

$$(\xi_{-1},\xi_1) \subset p(\mathbb{R}) \subset (\xi_{-2},\xi_2) \text{ and } (\xi_{-1},\xi_1) \subset q(\mathbb{R}) \subset (\xi_{-2},\xi_2).$$

Hence p and q are LSOP solutions of Eq. (1.1) with range in (ξ_{-2},ξ_2). Obviously we may assume that p and q are normalized. It was pointed out in Remark 3.20 that $x^{\Sigma(a^*(\varepsilon),\varepsilon)}(\mathbb{R}) \subsetneq x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}(\mathbb{R})$. Therefore $p(\mathbb{R}) \subsetneq q(\mathbb{R})$ provided δ_0 is small enough.

As we have seen in Section 2, one may define a C^1-smooth Poincaré return map P in a small neighborhood of p_0 in $H = \{\varphi \in C : \varphi(-1) = 0\}$ with fixed point p_0. As the Poincaré return map depends smoothly on the right side of the equation and as f is close to $f^{7,\varepsilon}$ in C_b^1-norm, we may suppose using Proposition 4.4 that $DP(p_0)$ has exactly two eigenvalues $\lambda_1 > \lambda_2 > 1$ with absolute value not smaller than 1. So $\mathcal{O}_p$ is hyperbolic with two Floquet multipliers outside the unit circle. Similarly, Proposition 4.4 implies $\mathcal{O}_q$ is hyperbolic with exactly one Floquet multiplier outside the unit circle. □

The statement of the previous proposition holds even if we consider functions in $C_b^1(D,\mathbb{R})$, where D is chosen to be any open set containing

$$\left\{x^{\Sigma(a^*(\varepsilon),\varepsilon)}(t) : t \in \mathbb{R}\right\} \cup \left\{x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}(t) : t \in \mathbb{R}\right\}.$$

In order to verify Theorem 1.1, we have to exclude the existence of more normalized LSOP solutions. The proof of Theorem 1.1 is completed at the end of Section 6.

5. Properties of Periodic Solutions

This section describes some useful properties of periodic solutions of Eq. (1.1). The next two results are well-known for the case when f is smooth and strictly increasing, see [17],[21] and [22]. The first proposition is analogous to Theorem 7.1 in [22] and the proof presented here is a slight modification of the proof of Theorem 7.1 in [22].

Proposition 5.1. (Monotonicity) *Assume that $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing and bounded, f is either continuously differentiable or there exist $u_1 < u_2 < \ldots < u_N$ with $N \geq 1$ so that $f|_{(-\infty,u_1]}$, $f|_{[u_1,u_2]}, \ldots, f|_{[u_N,\infty)}$ are continuously differentiable. If $p : \mathbb{R} \to \mathbb{R}$ is a nontrivial periodic solution of Eq.* (1.1)*, then p is of monotone type.*

Proof. Set points $t_0 < t_1 < t_0 + \omega$ so that $p(t_0) = \min_{t\in\mathbb{R}} p(t)$ and $p(t_1) = \max_{t\in\mathbb{R}} p(t)$, where ω is the minimal period of p. We have to verify that $\dot{p}(t) \geq 0$ for $t \in (t_0, t_1)$ and $\dot{p}(t) \leq 0$ for $t \in (t_1, t_0 + \omega)$.

To prove the lemma indirectly, assume that $\dot{p}(t) < 0$ for some $t \in (t_0, t_1)$.

Recall that ξ is a regular value of p, if for each $t \in \mathbb{R}$ with $p(t) = \xi$, $\dot{p}(t) \neq 0$ holds. According to Sard's Lemma [25], we may choose $\xi \in (p(t_0), p(t_1))$ so that ξ is a regular value of p and $p(t^*) = \xi$, $\dot{p}(t^*) < 0$ for some $t^* \in (t_0, t_1)$. Fix such ξ and t^*. Since p is continuously differentiable, one may give $t_2 \in (t_0, t^*)$ and $t_3 \in (t^*, t_1)$ so that $p(t_2) = p(t_3) = \xi$, $\dot{p}(t_2) > 0$, $\dot{p}(t_3) > 0$ and for $t \in (t_2, t_3) \setminus \{t^*\}$, $p(t) \neq \xi$.

Define the curves

$$\Gamma : [t_0, t_0 + \omega] \ni t \mapsto \pi p_t = (p(t), p(t-1)) \in \mathbb{R}^2$$

and

$$L : [0, 1] \ni s \mapsto (\xi, sp(t_2 - 1)) + (1 - s)\, p(t_3 - 1)) \in \mathbb{R}^2.$$

We claim that Γ is a simple closed curve. If not, then there exist t_4, t_5 with $t_0 \leq t_4 < t_5 < t_0 + \omega$ so that $\Gamma(t_4) = \Gamma(t_5)$. With $x(t) := p(t + t_4)$ and $\tilde{x}(t) := p(t + t_5)$, Proposition 2.4 implies $\pi x_0 \neq \pi \tilde{x}_0$, a contradiction. Thus curve Γ is simple.

Next we claim that if $t \in [t_0, t_1]$ with $p(t) = \xi$ and $\dot{p}(t) < 0$, then $\Gamma(t) \notin L$. Indeed, for such t we have $f(p(t-1)) = \dot{p}(t) + \xi < \xi$, while $f(p(t_i - 1)) = \dot{p}(t_i) + \xi > \xi$ for $i \in \{2, 3\}$. As f is monotone nondecreasing, the claim follows.

As a result, $J = \Gamma|_{(t_2,t_3)} \cup L$ is a simple closed curve.

Since $\dot{p}(t_2) > 0$, $\dot{p}(t_3) > 0$ and $\Gamma(t_2) \neq \Gamma(t_3)$, there exist $\varepsilon > 0$, C^1-maps $\gamma_j : [\xi - \varepsilon, \xi + \varepsilon] \to \mathbb{R}$, constants $\delta_j^+ > 0$, $\delta_j^- > 0$ for $j \in \{2, 3\}$ so that

$$\{(u, \gamma_j(u)) : u \in [\xi - \varepsilon, \xi + \varepsilon]\} = \Gamma\left(\left[t_j - \delta_j^-, t_j + \delta_j^+\right]\right), \quad j \in \{2, 3\}$$

and

$$\begin{aligned} R^- &= \{(u, v) : u \in (\xi - \varepsilon, \xi), v \text{ is in the open interval defined by } \gamma_2(u) \text{ and } \gamma_3(u)\} \\ R^+ &= \{(u, v) : u \in (\xi, \xi + \varepsilon), v \text{ is in the open interval defined by } \gamma_2(u) \text{ and } \gamma_3(u)\} \end{aligned}$$

belongs to different connected components of $\mathbb{R}^2 \setminus J$ (since $\Gamma(t) \notin L$ for all $t \in (t_2, t_3)$). We have $\Gamma\left(t_2 - \delta_2^-\right) \notin J$, $\Gamma\left(t_3 + \delta_3^+\right) \notin J$ and $\Gamma\left(t_2 - \delta_2^-\right) \in \overline{R^-}$, $\Gamma\left(t_3 + \delta_3^+\right) \in \overline{R^+}$.

Combining the above facts, we conclude that $\Gamma\left(t_2-\delta_2^-\right)$ and $\Gamma\left(t_3+\delta_3^+\right)$ belong to different connected components of $\mathbb{R}^2\backslash J$. Clearly, $\Gamma(t_0)$ and $\Gamma(t_1)$ belong to the exterior of J. Then in case $\Gamma\left(t_2-\delta_2^-\right)\in\operatorname{int}(J)$ there exists $t^{**}\in(t_0,t_2)$ such that Γ enters from $\operatorname{ext}(J)$ into $\operatorname{int}(J)$ through $\Gamma(t^{**})\in L$. In this case $R^+\subset\operatorname{ext}(J)$, $R^-\subset\operatorname{int}(J)$ and $\dot p(t^{**})<0$ follows. This is a contradiction to the fact that if $t\in[t_0,t_1]$ with $p(t)=\xi$ and $\dot p(t)<0$, then $\Gamma(t)\notin L$.

If $\Gamma\left(t_3+\delta_3^+\right)\in\operatorname{int}(J)$, then there is $t^{**}\in(t_3,t_1)$ so that Γ enters from $\operatorname{int}(J)$ into $\operatorname{ext}(J)$ through $\Gamma(t^{**})\in L$. We also have $R^+\subset\operatorname{int}(J)$, $R^-\subset\operatorname{ext}(J)$ in this case and again $\dot p(t^{**})<0$ follows, a contradiction.

The assumption that $\dot p(t)>0$ for some $t\in(t_1,t_0+\omega)$ leads to a contradiction analogously. □

The following statement resembles Theorem 7.2 in [22]. As we consider only scalar equations, the proof is elementary in our case.

Proposition 5.2. (Symmetry) *Assume the hypotheses of Proposition 5.1 and in addition suppose that $f(0)=0$, f is odd and 0 belongs to the range of p. Then p is of special symmetry.*

Proof. Let ω denote the minimal period of p. Set points $t_0<t_1<t_0+\omega$ as in the previous proof, that is with $p(t_0)=\min_{t\in\mathbb{R}}p(t)<0$ and $p(t_1)=\max_{t\in\mathbb{R}}p(t)>0$. According to Proposition 5.1, the set of zeros of p in (t_0,t_1) is an interval:

$$[z_{0,},z_1]=\{t\in(t_0,t_1):p(t)=0\}$$

with $t_0<z_0\le z_1<t_1$. Similarly, one may set z_2 and z_3 so that $[z_2,z_3]\subset(t_1,t_0+\omega)$, $p(t)=0$ for $t\in[z_{2,},z_3]$ and $p(t)\neq0$ for $t\in(t_1,t_0+\omega)\setminus[z_{2,},z_3]$. Of course, $z_0=z_1$ or $z_2=z_3$ is possible.

Consider the curve $\Gamma:[t_0,t_0+\omega]\ni t\mapsto\pi p_t\in\mathbb{R}^2$. As we have verified in the proof of Proposition 5.1, Γ is a simple closed curve. Setting $x=p$ and $\tilde x\equiv0$, Proposition 2.4 yields that $\Gamma(t)\neq(0,0)^{tr}$ for $t\in[t_0,t_0+\omega]$.

Next we verify that $(0,0)^{tr}\in\operatorname{int}(\Gamma)$. For $t\in(z_1,t_1]$, $p(t)>0$, $\dot p(t)\ge0$, hence $f(p(t-1))=\dot p(t)+p(t)>0$ and necessarily $p(t-1)>0$. We claim that $p(t-1)>0$ holds also for $t\in[z_{0,},z_1]$. If not, then there exists $z^*\in[z_{0,},z_1]$ so that $p(z^*-1)=0$, which contradicts $\Gamma(z^*)\neq(0,0)^{tr}$. Therefore

$$\Gamma(t)\in\left\{(u,v)\in\mathbb{R}^2:u\ge0,\,v>0\right\}\text{ for }t\in[z_0,t_1].$$

If $t\in(z_3,t_0+\omega]$, then $p(t)<0$, $\dot p(t)\le0$, hence $f(p(t-1))=\dot p(t)+p(t)<0$ and $p(t-1)<0$. It can be verified in a similar manner that $p(t-1)<0$ holds for $t\in[z_2,z_3]$

and thus

$$\Gamma(t) \in \left\{(u,v) \in \mathbb{R}^2 : u \le 0,\ v < 0\right\} \text{ for } t \in [z_2, t_0 + \omega].$$

Since Γ is a simple closed curve and there exists no $t \in [t_0, t_0+\omega] \setminus ([z_0, t_1] \cup [z_2, t_0+\omega])$ such that $\Gamma(t)$ is on the ordinate-axis, we obtain that $(0,0)^{tr} \in \text{int}(\Gamma)$.

Now take the periodic function $q : \mathbb{R} \ni t \mapsto -p(t) \in \mathbb{R}$ with minimal period ω and consider the curve $\Gamma' : [t_0, t_0+\omega] \ni t \mapsto \pi q_t \in \mathbb{R}^2$. Since f is odd, q is a solution of Eq. (1.1). Clearly $\Gamma'(t) = -\Gamma(t)$ for all $t \in [t_0, t_0+\omega]$. Because $(0,0)^{tr} \in \text{int}(\Gamma)$, curves Γ and Γ' intersect, that is $t^* \in [t_0, t_0+\omega]$ and $t^{**} \in [t_0, t_0+\omega]$ can be given with $\Gamma(t^*) = \Gamma'(t^{**})$. Set $\tilde{q} : \mathbb{R} \ni t \mapsto p(t + t^* - t^{**}) \in \mathbb{R}$. If q and $\tilde{q}$ are different periodic solutions of Eq. (1.1), then Proposition 2.4 implies $\pi q_{t^{**}} \neq \pi \tilde{q}_{t^{**}}$, that is $\Gamma(t^*) \neq \Gamma'(t^{**})$, a contradiction. So $p(t + t^* - t^{**}) = -p(t)$ for all t. Necessarily $t^* - t^{**} = \omega/2$. □

Note that we have an analogous result for special nonlinearity $f^{K,0}$; it is shown in Section 3 that for $K > K^*$, periodic solutions $x^{\Sigma(a^*,0)} : \mathbb{R} \to \mathbb{R}$ and $x^{\widetilde{\Sigma}(\tilde{a},0)} : \mathbb{R} \to \mathbb{R}$ of Eq. (3.1) are of monotone type and special symmetry. We conjecture that for case $\varepsilon = 0$, all nontrivial periodic solutions of Eq. (3.1) are in possession of these properties.

Let $K_0 > 3$ and $K_1 > K_0$ be fixed. Choose

$$\delta = \min_{K_0 \le K \le K_1} \frac{e^{1/K} - 1}{2(K+1)} > 0.$$

The next result is slightly more general than necessary in this paper. The stated property uniformly holds for K in a compact interval.

Proposition 5.3. *Assume $\mu = 1$, $K \in [K_0, K_1]$, $\varepsilon \in (0, \delta)$ and $p : \mathbb{R} \to \mathbb{R}$ is a normalized LSOP solution of Eq. (3.1) with minimal period $\omega > 0$. Then p is of monotone type and special symmetry, and the following assertions hold.*

(i) The zeros of p are simple.

(ii) $\omega \in \left(1 + \frac{1}{K}, 2 - \frac{1}{2K}\right)$.

(iii) $\max_{t \in \mathbb{R}} p(t) > e^{1/K}$.

(iv) Choose $t_{\max} \in (-1, 0)$ with $p(t_{\max}) = \max_{t\in\mathbb{R}} p(t)$. Let t_1 be the largest $t \in (-1, t_{\max})$ with $p(t) = 1$, and let t_4 be the smallest $t \in (t_{\max}, \infty)$ with $p(t) = 1$. Then

$$\dot{p}(t) \ge K - 2 \text{ for all } t \in (t_1 - \delta, t_1 + \delta),$$

$$\dot{p}(t) \le -\frac{1}{2} \text{ for all } t \in (t_4 - \delta, t_4 + \delta).$$

Let t_2 be the largest $t \in (t_1, t_{\max})$ with $p(t) = 1 + \varepsilon$, and let t_3 be the smallest $t \in (t_{\max}, t_4)$ with $p(t) = 1 + \varepsilon$.

(v) If $t_2+2<\omega$ and $t_4-t_1<1-\omega/2$, then $p_0\in\Sigma\left(U_\varepsilon^2,\varepsilon\right)$ with

$$p_0=\Sigma\left(t_3+2-\omega,t_1-t_4+\frac{\omega}{2},t_3-t_2,\varepsilon\right).$$

(vi) If $t_2+2<\omega$ and $t_3-t_1>1-\omega/2$, then $p_0\in\widetilde{\Sigma}\left(\widetilde{U}_\varepsilon^2,\varepsilon\right)$ with

$$p_0=\widetilde{\Sigma}\left(t_3+2-\omega,t_1-t_4+\frac{\omega}{2},t_3-t_2,\varepsilon\right).$$

Proof. Assume $p:\mathbb{R}\to\mathbb{R}$ is a normalized LSOP solution of Eq. (3.1). By definition, $V(p_t)=2$ for all $t\in\mathbb{R}$. Proposition 5.1 and Proposition 5.2 imply p is of monotone type and special symmetry. Setting $t_{\min}=t_{\max}+\omega/2$, we have $-p(t_{\max})=p(t_{\min})=\min_{t\in\mathbb{R}}p(t)$, p is monotone nondecreasing on intervals $[t_{\min}+k\omega,t_{\max}+(k+1)\omega]$, and monotone nonincreasing on intervals $[t_{\max}+k\omega,t_{\min}+k\omega]$, $k\in\mathbb{Z}$. By Proposition 2.4, $(p(t-1),p(t))\neq(0,0)$ for all $t\in\mathbb{R}$.

Claim (i). $\omega\in(1,2)$.

Proof. If $\omega\geq 2$, then $t_{\min}=t_{\max}+\omega/2>-1+\omega/2>0$. By the special symmetry, $p(-1+\omega/2)=p(-1)=0$. The monotone property yields $p(s)\geq 0$ for $s\in[-1,-1+\omega/2]$. Consequently $V(p_0)=0$, a contradiction. Suppose $\omega\leq 1$. Then $-1<t_{\max}<t_{\min}<-1+\omega\leq 0$, and $p(-1+\omega)=0$, $p(-1+\omega+s)>0$ for all $s\in(0,\eta)$ for some $\eta>0$. Clearly there is an arbitrarily small $s>0$ with $\dot{p}(-1+\omega+s)>0$. Then from Eq. (3.1)

$$f^{K,\varepsilon}(p(-1+\omega+s-1))=\dot{p}(-1+\omega+s)+p(-1+\omega+s)>0$$

and $p(-1+\omega+s-1)>1$ follow. By continuity, $p(-2+\omega)\geq 1$. Hence, by using $-2+\omega\leq -1$, $p(-1)=0$ and the monotone property of p, one obtains $\mu\in(-2+\omega,-1)$ with $p(\mu)<0$. Then p has at least 3 sign changes on $[-2+\omega,-1+\omega]$, a contradiction. Therefore $1<\omega<2$. □

Claim (ii). $p(0)<0$.

Proof. The equality $p(0)=0$ contradicts Proposition 2.4 since $p(-1)=0$. If $p(0)>0$, then by (3.1) and $p(-1)=0$, $\dot{p}(0)<0$. The monotone property of p yields either $p(s)\geq 0$ for all $s\in[-1,0]$ or $\omega<1$, a contradiction. Thus $p(0)<0$. □

From $p(0)<0$, by (3.1) and $p(-1)=0$, $\dot{p}(0)>0$ follows. Hence $t_{\min}<0$.

Set $\tau=\omega-1\in(0,1)$. It is easy to see that $p(t)\leq 0$ for all $t\in[0,\tau]$, and $p(t)>0$ for all $t\in(\tau,\tau+\eta)$ for some $\eta>0$.

Define $t_5 = t_1 + \omega/2$, $t_8 = t_4 + \omega/2$, $t_9 = t_1 + \omega$ and $t_{12} = t_4 + \omega$ (see Fig. 9). Clearly, $t_9 > \tau$. Note that $z = -1 + \omega/2 = \frac{\tau-1}{2} \in (t_4, t_5)$ is also a negative zero of p.

Observe that $0 < \varepsilon < \delta$ implies

$$\varepsilon < \frac{1}{K} \text{ and } \varepsilon < \frac{e^{\frac{1}{K}}}{2K}.$$

Claim (iii). $\tau \in (t_1 + 1, t_4 + 1)$.

Proof. As $p(0) < 0$ and p is of type (0) on $[0, t_1 + 1]$,

$$p(t) = p(0)e^{-t} < 0 \text{ for all } t \in [0, t_1 + 1]\,. \tag{5.1}$$

So $\tau > t_1 + 1$. If $p(t_4 + 1) < 0$, then on the one hand $p(t) < 0$ for all $t \in [t_4 + 1, z + 1]$ (as p is of type (0) on $[t_4 + 1, z + 1]$), on the other hand

$$z + 1 = \frac{\omega}{2} \in \left(\tau, \tau + \frac{\omega}{2}\right)$$

and $p(t) \geq 0$ for all $t \in [\tau, \tau + \omega/2]$, a contradiction. If $p(t_4 + 1) = 0$, then $p(t) = 0$ for all $t \in [t_4 + 1, z + 1]$. By $(\tau + 1)$-periodicity, $p(t) = 0$ follows for $t \in [t_4 - \tau, z - \tau]$. By the definitions of t_1, t_4 and z, the minimal zero of p in $(-1, z]$ is in $(t_4, z]$. As $z = (\tau - 1)/2 > \tau - 1$ and thus $z - \tau > -1$, this a contradiction. Consequently, $p(t_4 + 1) > 0$ and $\tau \in (t_1 + 1, t_4 + 1)$. □

Proofs of Assertions (i) and (ii). Assertion (i) is a direct consequence of Claim (iii). Note that if $t \in (t_1 + 1, t_4 + 1)$ with $p(t) = 0$, then

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) = f^{K,\varepsilon}(p(t-1)) > 0.$$

Hence τ is a simple zero of p, and it is the only zero in $(t_1 + 1, t_4 + 1)$. By the special symmetry of p, all zeros of p are simple, and $-1, z, \tau$ are the only zeros in $[-1, \tau]$.

Assertion (ii) also follows from Claim (iii). Indeed, for $t \in [\tau, t_9]$,

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \leq f^{K,\varepsilon}(p(t-1)) \leq K.$$

Hence

$$t_9 - \tau = t_1 + \omega - \tau \geq 1/K. \tag{5.2}$$

Applying (5.2) and $\tau > t_1 + 1$, we get

$$\omega \geq \tau - t_1 + \frac{1}{K} = \tau - (t_1 + 1) + 1 + \frac{1}{K} > 1 + \frac{1}{K}.$$

For all $t \in \mathbb{R}$, $|p(t)| \le K$ by Proposition 2.6, and thus $\dot{p}(t) \ge -2K$ by Eq. (3.1). Hence

$$t_4 \le z - \frac{1}{2K} < -\frac{1}{2K},$$

and by Claim (iii),

$$\omega = \tau + 1 < t_4 + 2 < 2 - \frac{1}{2K}.$$

□

Claim (iv). $\max_{t\in\mathbb{R}} p(t) \ge e^{1/K}$.

Proof. We have already shown that $p(t_4+1) > 0$. For $t \in [t_4+1, t_5+1]$, $p(t) = p(t_4+1)\, e^{-(t-t_4-1)}$, thus p strictly decreases on $[t_4+1, t_5+1]$. So $t_9 < t_4+1$. As $t_4 + 1 + 1/K < t_4 + \omega = t_{12}$, we derive that $[t_4+1, t_4+1+1/K] \subset [t_9, t_{12})$ and thus $p(t_4+1+1/K) \ge 1$.

From (5.2), $t_5 - t_4 \ge 1/K$ follows by special symmetry. So p is of type (0) on $[t_4+1, t_4+1+1/K]$ and thus

$$\max_{t\in\mathbb{R}} p(t) \ge p(t_4+1) = p\left(t_4 + 1 + \frac{1}{K}\right) e^{\frac{1}{K}} \ge e^{\frac{1}{K}}.$$

□

As a consequence of Claim (iv), $\max_{t\in\mathbb{R}} p(t) > 1 + 1/K > 1 + \varepsilon$ and one may set t_2 and t_3, so that t_2 is the maximal $t \in (t_1, t_{\max})$ with $p(t) = 1+\varepsilon$ and t_3 is the minimal $t \in (t_{\max}, t_4)$ with $p(t) = 1+\varepsilon$. Define $t_6 = t_2 + \omega/2$, $t_7 = t_3 + \omega/2$, $t_{10} = t_2 + \omega$ and $t_{11} = t_3 + \omega$ (see Fig. 9).

Note that it is also verified in the proof of the previous claim that

$$t_{12} - (t_4 + 1) > \frac{1}{K}. \tag{5.3}$$

Claim (v). $\dot{p}(t) \le -1$ for $t \in [t_4+1, t_{12}]$, and thus $t_4 - t_3 = t_{12} - t_{11} \le \varepsilon$.

Proof. First note that $t_{12} < \tau + \omega/2 < \tau + 1$. Hence for $t \in [t_4+1, t_{12}]$, $p(t) \ge 1$, $p(t-1) \le 1$, and

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \le -p(t) \le -1,$$

which is our first assertion. In addition, using $p(t_{12}) = 1$ and estimation (5.3), we obtain that $p(t_{12} - s) \ge 1 + s$ for all $0 \le s \le 1/K$. Hence $t_{12} - t_{11} \le \varepsilon$. □

Claim (vi). $1 + t_2 < t_9$ and $t_{10} < t_3 + 1$. In consequence, $t_2 - t_1 \le \varepsilon/(K-2)$.

Proof. It follows from the previous claim that $p(t) \geq 0$ for all $t \in [t_3 + 1, t_4 + 1]$. Indeed, $p(t) \geq p(t_4 + 1) - 2K\varepsilon > e^{1/K} - 2K\varepsilon > 0$ for all $t \in [t_3 + 1, t_4 + 1]$ because $t_4 - t_3 \leq \varepsilon$, $p(t_4 + 1) \geq e^{\frac{1}{K}}$ and $\dot{p}(t) \geq -2K$ for all $t \in \mathbb{R}$. Hence $\dot{p}(t) \leq K$ for $t \in [t_3 + 1, t_4 + 1]$.

Suppose that $t_3 + 1 \leq t_{10}$, that is $p(t_3 + 1) \leq 1 + \varepsilon$. Applying the facts that $t_4 - t_3 \leq \varepsilon$ and p strictly decreases on $[t_4 + 1, t_{12}]$ (see Claim (v)), we obtain that

$$\max_{t \in \mathbb{R}} p(t) = \max_{t \in [t_3+1, t_4+1]} p(t) \leq 1 + \varepsilon + K\varepsilon = 1 + (K+1)\varepsilon < e^{\frac{1}{K}}$$

by $\varepsilon \in (0, \delta)$, a contradiction to Claim (iv). Thus $t_{10} < t_3 + 1$.

If $t_9 \leq t_2 + 1$, then

$$t_9 \leq t_2 + 1 < t_2 + 1 + \frac{1}{K} < t_2 + \omega = t_{10} < t_3 + 1$$

and hence for $t \in [t_2 + 1, t_2 + 1 + 1/K]$,

$$\dot{p}(t) = -p(t) + K \geq -(1+\varepsilon) + K.$$

Thus

$$1 + \varepsilon = p(t_{10}) \geq p\left(t_2 + \frac{K+1}{K}\right) \geq p(t_2 + 1) + \frac{K - 1 - \varepsilon}{K} \geq 1 + \frac{K - 1 - \varepsilon}{K},$$

which contradicts $\varepsilon < \delta$. So $1 + t_2 < t_9$.

As a result, $\dot{p}(t) = -p(t) + K \geq K - 2$ for all $t \in [t_9, t_{10}]$, and $\varepsilon = \int_{t_9}^{t_{10}} \dot{p}(s)\, ds \geq (K-2)(t_{10} - t_9)$. As $t_2 - t_1 = t_{10} - t_9$, the third statement follows. □

Claim (vii). $t_9 - (t_2 + 1) > \delta$ and $t_3 + 1 - t_9 > \delta$.

Proof. Applying $p(t_1 + 1) < 0$ by (5.1), $t_2 - t_1 \leq \varepsilon/(K-2)$ by Claim (vi) and $\dot{p}(t) \leq 2K$ for all $t \in \mathbb{R}$, we find $p(t_2 + 1) < 2K\varepsilon/(K-2)$. Therefore

$$1 - \frac{2K}{K-2}\varepsilon < p(t_9) - p(t_2 + 1) = \int_{t_2+1}^{t_9} \dot{p}(s)\, ds \leq 2K(t_9 - t_2 - 1),$$

and we obtain that

$$t_9 - (t_2 + 1) > \frac{1}{2K} - \frac{\varepsilon}{K-2} > \delta. \tag{5.4}$$

Claim (iv) implies that $\max p(t) \geq e^{1/K}$. Claim (vi) gives that $t_9 < t_3 + 1 < t_4 + 1 < t_{12}$. For $t \in [t_9, t_4 + 1]$,

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \leq -p(t) + K \leq K - 1.$$

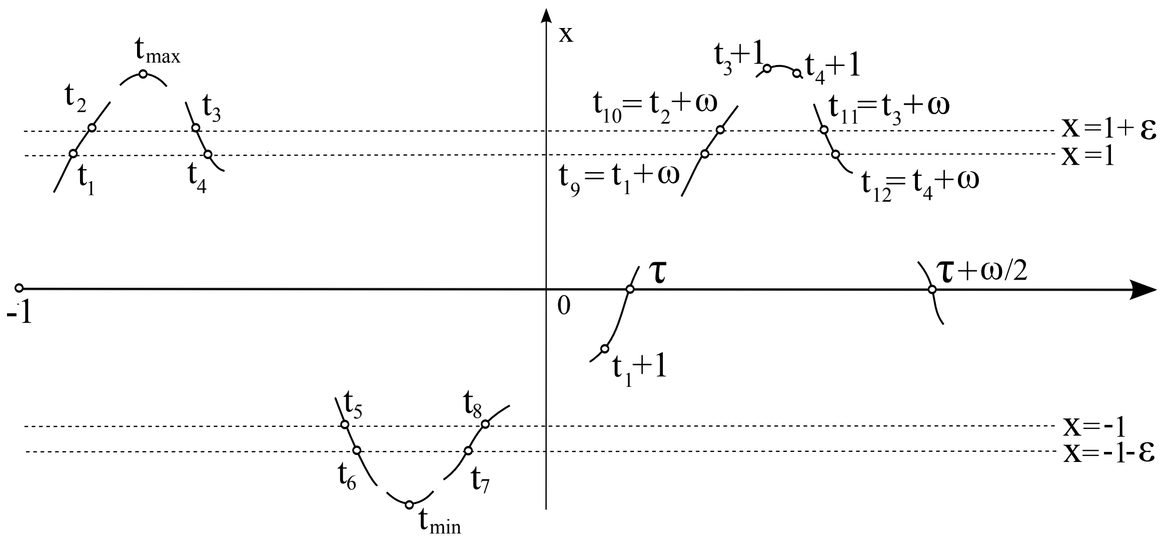

FIGURE 9. The plot of p in the proof of Proposition 5.3

In addition, p strictly decreases on $[t_4+1, t_{12}]$ by Claim (v), that is $\max_{t\in\mathbb{R}} p(t) = \max_{t\in[t_9,t_4+1]} p(t)$. So, by using Claim (v) again, we obtain that

$$\begin{aligned} e^{\frac{1}{K}} \le \max_{t\in\mathbb{R}} p(t) = \max_{t\in[t_9,t_4+1]} p(t) &\le 1+(K-1)(t_4+1-t_9) \\ &= 1+(K-1)(t_3+1-t_9+t_4-t_3) \\ &\le 1+(K-1)(t_3+1-t_9)+(K-1)\varepsilon, \end{aligned}$$

from which

$$t_3+1-t_9 \ge \frac{e^{1/K}-1}{K-1} - \varepsilon > \frac{e^{1/K}-1}{2(K+1)} \ge \delta \tag{5.5}$$

follows. □

Proof of Assertion (iv). Note that if $t \ge t_{12}$, then $\dot{p}(t) \ge -2K$ and

$$p(t) \ge 1-2K(t-t_{12}).$$

Thus

$$p(t) \ge 1/2 \text{ for all } t \in \left[t_{12}, t_{12}+\frac{1}{4K}\right]. \tag{5.6}$$

Then Claim (vii) implies that for $t \in (t_9-\delta, t_9+\delta)$, $t-1 \in (t_2, t_3)$. Also, $p(t) \le p(t_9)+\delta \max \dot{p}(t) \le 1+2K\delta \le 2$ for $t \in (t_9-\delta, t_9+\delta)$. Hence

$$\dot{p}(t) = -p(t)+f^{K,\varepsilon}(p(t-1)) \ge -2+K \text{ for } t \in (t_9-\delta, t_9+\delta).$$

As $t_{12}-(t_4+1)=t_4+\omega-(t_4+1)>1/K>\delta$, Claim (v) and $p(t_{12})=1$ imply that $p(t)\geq 1$ and $\dot{p}(t)\leq -\frac{1}{2}$ for $t\in(t_{12}-\delta,t_{12}]$. It was pointed out in the proof of Claim (iv) that $t_4+1/K\leq t_5<0$. In consequence,

$$t_{12}+\delta-1=(t_4+1+\tau)+\delta-1=\tau+t_4+\delta\leq\tau+t_5<\tau$$

that is $t-1\in(t_4,\tau)$ for all $t\in[t_{12},t_{12}+\delta)$. Thus from (5.6) we conclude that

$$\dot{p}(t)=-p(t)+f^{K,\varepsilon}(p(t-1))\leq-\frac{1}{2}+0=-\frac{1}{2},\quad t\in[t_{12},t_{12}+\delta).$$

Statement (iv) follows by periodicity. □

Proofs of Assertions (v) and (vi). Suppose $t_2+2<\omega$ and $t_4<t_1+1-\omega/2$. Then $t_2+1<\omega-1=\tau$, $t_{10}<t_9+\delta<t_3+1$ and

$$t_4+1<t_{11}<t_{12}=t_4+\omega<t_1+1+\omega/2=t_5+1.$$

It follows that p is of type (K) on $[\tau,t_3+1]$, it is of type (0) on $[t_4+1,t_5+1]$. The periodicity of p implies that p is of type (0) on $[t_3,t_4]$. Therefore p is of type $(0,1+\varepsilon)$ on $[t_3+1,t_4+1]$. By periodicity and special symmetry, p is of type $(-K)$ on $[t_5,t_6]$, and it is of type $(-K,-1)$ on $[t_5+1,t_6+1]$. The special symmetry and monotonicity yield $p_0=p_\omega\in\Sigma(U^2_\varepsilon,\varepsilon)$ with

$$p_0=\Sigma\left(t_3+2-\omega,t_1-t_4+\frac{\omega}{2},t_3-t_2,\varepsilon\right).$$

The case $t_2+2<\omega$ and $t_3-t_1>1-\omega/2$ is analogous. Note that under these conditions $t_2+1<\tau$, $t_{10}<t_3+1$ and $t_5+1<t_{11}$. □

6. On the Number of Periodic Solutions

Set $\mu=1$. We study the exact number of LSOP solutions of Eq. (1.1) first for nonlinearity $f^{K,0}$ with $K>0$, then for $f^{7,\varepsilon}$ with $\varepsilon>0$ small, finally for those feedback functions, that are close to $f^{7,\varepsilon}$ in C^1_b-norm. As a consequence, we prove Theorem 1.1.

For simplicity, we use notations introduced in Section 3 - without repeating definitions.

6.1. **The number of periodic solutions for the step function.** As a preliminary result, we show that K has to be sufficiently large so that Eq. (3.1) has periodic solutions of monotone type and special symmetry.

Proposition 6.1. *Suppose $K>0$, $\varepsilon\in[0,1)$, $p:\mathbb{R}\to\mathbb{R}$ is a nontrivial periodic solution of Eq.* (3.1)*, and p is of monotone type and special symmetry. Then $K>1$*

and

$$\frac{\omega}{2} \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K},$$

where $\omega > 0$ denotes the minimal period of p.

Proof. Let $p : \mathbb{R} \to \mathbb{R}$ be a periodic solution of Eq. (3.1) of monotone type and special symmetry and with minimal period $\omega > 0$. It is no restriction to assume that p is normalized. Then $p(-1) = p(-1 + \omega/2) = 0$. Clearly, $\max_{t \in \mathbb{R}} p(t) > 1$ as $\dot{x}(t) = -x(t)$ has no periodic solutions. Then there exists $(c_1, c_2, c_3) \in (0,1)^3$ with $c_1 + c_2 + c_3 = \omega/2$ so that $p(-1 + c_1) = p(-1 + c_1 + c_2) = 1$, p is nondecreasing on $[-1, -1 + c_1]$ with range $[0,1]$, $p(t) > 1$ for $t \in (-1 + c_1, -1 + c_1 + c_2)$ and p is nonincreasing on $[-1 + c_1 + c_2, -1 + \omega/2]$ with range $[0,1]$. The choice of constants c_i and the special symmetry of p imply the following: if $p(t) > 1$ for all $t \in I$, where $I \subset \mathbb{R}$ is an interval, then the length of I is not greater than c_2.

As $\dot{p}(t) \geq 0$ almost everywhere on $[-1, -1 + c_1]$,

$$f^{K,\varepsilon}(p(t-1)) = \dot{p}(t) + p(t) > 0 \text{ for } t \in (-1, -1 + c_1],$$

that is $p(t) > 1$ for $t \in (-2, -2 + c_1]$. We conclude that $c_2 \geq c_1$.

Obviously, $(e^t p(t))' = e^t f^{K,\varepsilon}(p(t-1))$ almost everywhere on $\mathbb{R}$. Integrating on $[-1, -1 + c_1]$, we get

$$e^{-1+c_1} = \int_{-1}^{-1+c_1} e^s f^{K,\varepsilon}(p(s-1))\,\mathrm{d}s \leq K \int_{-1}^{-1+c_1} e^s \mathrm{d}s = K\left[e^{-1+c_1} - e^{-1}\right],$$

thus $1 \leq K(1 - e^{-c_1})$. As $1 - e^{-c_1} < 1$, necessarily $K > 1$ and $c_1 \geq \ln(K/(K-1))$. Integrating on $[-1 + c_1 + c_2, -1 + c_1 + c_2 + c_3]$, we obtain that

$$-e^{-1+c_1+c_2} \geq -K \int_{-1+c_1+c_2}^{-1+c_1+c_2+c_3} e^s \mathrm{d}s = -K\left[e^{-1+c_1+c_2+c_3} - e^{-1+c_1+c_2}\right],$$

hence $1 \leq K(e^{c_3} - 1)$ and $c_3 \geq \ln((K+1)/K)$.

Therefore

$$\frac{\omega}{2} = c_1 + c_2 + c_3 \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K}.$$

□

Corollary 6.2. *For $K \in (0,3]$ and $\varepsilon = 0$, Eq. (3.1) admits no LSOP solutions.*

Proof. It is excluded by the previous proposition that we have LSOP solutions for $K \in (0,1]$ and $\varepsilon = 0$. Suppose $K \in (1,3]$, $\varepsilon = 0$ and $p : \mathbb{R} \to \mathbb{R}$ is an LSOP solution of Eq. (3.1) with minimal period $\omega < 2$. Then Proposition 6.1 yields that

$$1 > \frac{\omega}{2} \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K} = \ln \frac{K(K+1)}{(K-1)^2},$$

that is

$$e > \frac{K(K+1)}{(K-1)^2} = 1 + \frac{3}{K-1} + \frac{2}{(K-1)^2}.$$

This is a second order inequality for $z = 1/(K-1)$, hence the solution formula gives that

$$z_1 = \frac{-3-\sqrt{8e+1}}{4} < \frac{1}{K-1} < z_2 = \frac{-3+\sqrt{8e+1}}{4}.$$

The first inequality is clearly satisfied as $K > 1$ and $z_1 < 0$. The second inequality implies $K > 1 + 1/z_2 > 3$, a contradiction. □

Recall from Remarks 3.4 and 3.14 that $\varphi \in C$ is in $\Sigma(U_0^1, 0) = \widetilde{\Sigma}\left(\widetilde{U}_0^1, 0\right)$ if and only if $\varphi(-1) = 0$ and there there exist $-1 < s_1 < s_2 < s_3 < 0$ so that x^φ is of type (K) on $[-1, s_1]$, of type (0) on $[s_1, s_2]$, of type $(-K)$ on $[s_2, s_3]$ and of type (0) on $[s_3, 0]$.

Proposition 6.3. *Assume $K > 3$, $\varepsilon = 0$ and $x : \mathbb{R} \to \mathbb{R}$ is a nontrivial periodic solution of Eq.* (3.1)*, x is of special symmetry and $x_0 \in \Sigma(U_0^1, 0) = \widetilde{\Sigma}\left(\widetilde{U}_0^1, 0\right)$. Then $x(s_2) = 1$ implies $K = K^*$.*

Proof. Assume that x satisfies the conditions of the proposition with $x(s_2) = 1$.

Then using (3.4) and the definitions of I_1 and I_2, we get

$$x(s_1) = e^{-s_1} I_1 = K\left(1 - e^{-a_1}\right) \tag{6.1}$$

and

$$e^{a_2} = e^{a_2} x(s_2) = e^{a_2} e^{-s_2} I_2 = K\left(1 - e^{-a_1}\right). \tag{6.2}$$

From (3.4), the definition of I_3 and relation (6.2) it follows that

$$x(s_3) = e^{-s_3} I_3 = K\left(1 - e^{-a_1}\right) e^{-a_2-a_3} + K\left(e^{-a_3} - 1\right) = e^{-a_3} + K\left(e^{-a_3} - 1\right). \tag{6.3}$$

Let $-1 < t_1 = t_2 < t_3 = t_4 < \ldots$ be the consecutive times for which $x(t_i) \in \{-1, 1\}$. As x strictly increases on $[-1, s_1]$, strictly decreases on $[s_1, s_2]$, $\max_{t\in\mathbb{R}} x(t) > 1$ and $x(s_2) = 1$, we obtain that $-1 < t_1 < s_1$ and $t_3 = s_2$. Similarly, $s_2 < t_5 < s_3$. By special symmetry, $x(s_3) = -x(s_1)$, and $x(s_2) = -x(t_1 + 1) = 1$. So combining (6.1) and (6.3), we get

$$e^{a_3} = \frac{K+1}{K} e^{a_1}. \tag{6.4}$$

As in the proof of Proposition 3.3, we can show that

$$x(t_1 + 1) = \frac{K-1}{K} I_3.$$

Using $x(t_1+1)=-1$, the definition of I_3, relations (6.2) and (6.4), it follows that

$$-1=\frac{K-1}{e}\left[e^{a_1}-1+e^{a_1+a_2}\left(1-e^{a_3}\right)\right]=\frac{-1}{e}\left(K^2-1\right)\left(e^{a_1}-1\right)^2. \tag{6.5}$$

As x is periodic, $a_2=t_5-t_4$ (see the remark preceding Proposition 3.6). One may show analogously to the proof of Proposition 3.6 that

$$a_2=t_5-t_4=\ln\frac{I_2+Ke^{s_2}}{(K-1)\,I_1}=\ln\frac{e^{a_1}-1+e^{a_1+a_2}}{(K-1)\left(e^{a_1}-1\right)}.$$

Combining this relation with (6.2), we get that a_1 is the following function of K:

$$a_1=\ln\frac{K(K-1)}{K^2-2K-1}.$$

Substituting the last result to (6.5), we obtain that equation

$$\frac{\left(K^2-1\right)(K+1)^2}{\left(K^2-2K-1\right)^2}=e$$

holds for K, which equation has a unique solution on $(3,\infty)$ and that is K^* (see the definition of K^* before Proposition 3.8). So $K=K^*$. □

Proposition 6.4. *Assume $K\in(3,\infty)\setminus\{K^*\}$, $\varepsilon=0$ and $x:\mathbb{R}\to\mathbb{R}$ is a normalized LSOP solution of Eq. (3.1). Then $K>K^*$, and either $x_0=\Sigma(a^*,0)$, or $x_0=\widetilde{\Sigma}(\widetilde{a},0)$, where $\Sigma(a^*,0)$ and $\widetilde{\Sigma}(\widetilde{a},0)$ are given in Section 3.*

Proof. Let τ denote the smallest zero of x on $[0,\infty)$ with the property that $x>0$ on $(\tau,\tau+\eta)$ with some $\eta>0$ small. Since x is normalized periodic solution with minimal period in $(1,2)$, and as it is of special symmetry and of monotone type, the minimal period is $\omega=\tau+1$ and $x(0)\le 0$.

Set $t_{\max}\in(-1,0)$ so that $x(t_{\max})=\max_{t\in\mathbb{R}}x(t)$ and choose $t_{\min}=t_{\max}+\omega/2$. Clearly $x(t_{\min})=\min_{t\in\mathbb{R}}x(t)$. As equation $\dot{x}(t)=-x(t)$ has no periodic solution, $x(t_{\max})=-x(t_{\min})>1$.

As x is of monotone type and $x(t_{\max})=-x(t_{\min})>1$, there exists $t_1\in(-1,t_{\max})$ maximal with $x(t)=1$ and $t_3\in(t_{\max},t_{\min})$ minimal with $x(t)=1$. Then $t_5=t_1+\omega/2$ is the maximal $t\in(t_{\max},t_{\min})$ with $x(t)=-1$ and $t_7=t_3+\omega/2$ is the minimal $t\in(t_{\min},\tau)$ with $x(t)=-1$.

Solution x must be piecewise of type (i) with $i\in\{-K,0,K\}$. To be more precise, x is of type (0) on interval $[0,t_1+1]$, of type (K) on $[t_1+1,t_3+1]$, of type (0) on $[t_3+1,t_5+1]$, of type $(-K)$ on $[t_5+1,t_7+1]$ and of type (0) on $[t_7+1,\tau+1]$. If $\tau<t_3+1$, then $(t_3+1-\tau,t_5-t_3,t_7-t_5)\in(0,1)^3$ and

$$(t_3+1-\tau)+(t_5-t_3)+(t_7-t_5)=t_7+1-\tau<1,$$

therefore

$$(t_3+1-\tau, t_5-t_3, t_7-t_5, 0) \in U^1 = \widetilde{U}^1.$$

If condition $t_1+1<\tau$ is satisfied besides $\tau<t_3+1$, then

$$x_0 = x_{\tau+1} = \Sigma\left(t_3+1-\tau, t_5-t_3, t_7-t_5, 0\right) = \widetilde{\Sigma}\left(t_3+1-\tau, t_5-t_3, t_7-t_5, 0\right)$$

by Remarks 3.4 and 3.14.

So we claim that $\tau \in [t_1+1, t_3+1)$. As x is of type (0) on $[0, t_1+1]$, $x(t) = x(0)e^{-t} \leq 0$ for $t \in [0, t_1+1]$. So $\tau \geq t_1+1$. Suppose for contradiction that $x(t_3+1) \leq 0$. Since $\dot{x}(t) \leq -x(t) + K$ for almost all $t \geq -1$ and $x(-1) = 0$, estimate

$$x(t) \leq K\left(1-e^{-(t+1)}\right) < K$$

holds for all $t \geq -1$. Then as

$$\dot{x}(t) = -x(t) + K > 0, \quad t_1+1 < t < t_3+1, \tag{6.6}$$

and as

$$x(t) = x(t_3+1)\, e^{-(t-t_3-1)}, \quad t_3+1 \leq t \leq t_5+1, \tag{6.7}$$

we get that x is nondecreasing and nonpositive on $[t_1+1, t_5+1]$. So $x(t) \leq 0$ on $[t_5, t_5+1]$. On the other hand, for $t_5+\omega/2 \in [t_5, t_5+1]$ we have $x(t_5+\omega/2) = x(t_1+\omega) = 1$, a contradiction. Thus $x(t_3+1) > 0$, $\tau \in [t_1+1, t_3+1)$ and $x_{\tau+1} \in \Sigma\left(U_0^1, 0\right) = \widetilde{\Sigma}\left(\widetilde{U}_0^1, 0\right)$.

Equations (6.6) and (6.7) now imply that x strictly increases on $[t_1+1, t_3+1]$ and strictly decreases on $[t_3+1, t_5+1]$. Thus $x(t_3+1)$ is a local maximum of x. As x is of monotone type and $\max_{t\in\mathbb{R}} x(t) > 1$, $x(t_3+1) > 1$ follows. Also, $x(t_5+1) > 0$ by (6.7). By special symmetry,

$$x(t_7+1) = x(t_3+\omega/2+1) = -x(t_3+1) < -1.$$

Remarks 3.4 and 3.14 yield that if $x(t_5+1) < 1$, then $x_0 = x_{\tau+1} \in \Sigma\left(U_0^2, 0\right)$; if $x(t_5+1) > 1$, then $x_0 = x_{\tau+1} \in \widetilde{\Sigma}\left(\widetilde{U}_0^2, 0\right)$. The case $x(t_5+1) = 1$ is excluded by Proposition 6.3.

We have already verified that $x(t_1+1) < 0$ and $x(t_3+1) > 0$. If $x_0 \in \Sigma\left(U_0^2, 0\right)$, then $x(t_1+1) = -x(t_5+1) > -1$, so Remark 3.4 yields that $x_0 \in \Sigma\left(U_0^3, 0\right)$ and thus $(t_3+1-\tau, t_5-t_3, t_7-t_5)$ is a fixed point of $F(\cdot, 0)$. Proposition 3.8 implies $K > K^*$ and $x_0 = \Sigma(a^*, 0)$. Similarly, if $x_0 \in \widetilde{\Sigma}\left(\widetilde{U}_0^2, 0\right)$, then $x_0 \in \widetilde{\Sigma}\left(\widetilde{U}_0^3, 0\right)$. By Proposition 3.18, $K > K^*$ and $x_0 = \widetilde{\Sigma}(\widetilde{a}, 0)$. □

As a direct consequence of Corollary 6.2 and Proposition 6.4, we get the following.

Theorem 6.5. *For $K \in (0, K^*)$ and $\varepsilon = 0$, Eq. (3.1) has no LSOP solutions. For $K > K^*$ and $\varepsilon = 0$, there are exactly two normalized LSOP solutions of Eq. (3.1).*

It can be also shown that in case $K = K^*$ and $\varepsilon = 0$, there is exactly one normalized LSOP solution.

6.2. **There are two LSOP solutions for $f^{7,\varepsilon}$ with $\varepsilon > 0$, and for close nonlinearities.** Recall that if $K = 7$ and $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$, where ε_0 and $\widetilde{\varepsilon}_0$ are given by Propositions 3.11 and 3.18, respectively, then Eq. (3.1) admits two LSOP solutions with initial functions $\Sigma(a^*(\varepsilon), \varepsilon)$ and $\widetilde{\Sigma}(\widetilde{a}(\varepsilon), \varepsilon)$.

Proposition 6.6. *Let $K = 7$. A threshold number $\varepsilon_* \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ can be given so that for $\varepsilon \in (0, \varepsilon_*)$, $x^{\Sigma(a^*(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ and $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)} : \mathbb{R} \to \mathbb{R}$ are the only normalized LSOP solutions of Eq. (3.1).*

Proof. Suppose for contradiction that there exists a sequence $(\varepsilon^n)_1^\infty$ in $(0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ converging to 0 as $n \to \infty$ and a sequence of functions $(x^n)_0^\infty$ so that for $n \geq 0$, $x^n : \mathbb{R} \mapsto \mathbb{R}$ is a normalized LSOP solution of (3.1) with $K = 7$ and $\varepsilon = \varepsilon^n$, and

$$x_0^n \notin \left\{ \Sigma(a^*(\varepsilon^n), \varepsilon^n), \widetilde{\Sigma}(\widetilde{a}(\varepsilon^n), \varepsilon^n) \right\}.$$

Let $\omega_n > 0$ denote the minimal period of x^n. According to Proposition 5.3 (ii), $\omega_n \in (8/7, 27/14)$ for all sufficiently large n.

For all $t \in \mathbb{R}$ and $n \in \mathbb{N}$, Proposition 2.6 implies $|x^n(t)| \leq 7$, therefore Eq. (3.1) gives $|\dot{x}^n(t)| \leq 14$. Applying the Arzelà–Ascoli theorem and changing to a subsequence if necessary, we may assume that there is $\omega \in [8/7, 27/14]$ and a continuous function $x : \mathbb{R} \to \mathbb{R}$ such that $\omega_n \to \omega$ as $n \to \infty$, and $x^n(t) \to x(t)$ as $n \to \infty$ uniformly on all compact subsets of the real line. It is easy to see that x is periodic with minimal period ω, it is of monotone type and special symmetry. In addition, $x(-1) = x(-1 + \omega/2) = 0$ and $x(t) \geq 0$ for $t \in [-1, -1 + \omega/2]$. By Proposition 5.3 (iii),

$$\max_{t \in \mathbb{R}} x(t) \geq \liminf_{n \to \infty} \max_{t \in \mathbb{R}} x^n(t) \geq e^{\frac{1}{7}}.$$

Proposition 5.3 (iv) gives that if $t_0 \in \mathbb{R}$ and $|x(t_0)| = 1$, then

$$\liminf_{h \to 0} \left| \frac{x(t+h) - x(t)}{h} \right| \geq \frac{1}{2} \text{ for all } t \in (t_0 - \delta, t_0 + \delta)$$

with constant $\delta > 0$ defined before Proposition 5.3. Therefore there exist unique $t_1, t_4 \in [-1, -1 + \omega/2]$ with $-1 < t_1 < t_4 < -1 + \omega/2$ such that $x(t_1) = x(t_4) = 1$. In addition, for all $\gamma \in (0, \delta/2)$ fixed, $|x(t) - 1| \geq \gamma$ for all $t \in [-1, -1 + \omega/2]$ with $|t - t_1| \geq 2\gamma$

and $|t - t_4| \geq 2\gamma$. Set

$$S_\gamma = \{s \in [-1, 0] : \ x(s) \in (-1-\gamma, -1+\gamma) \cup (1-\gamma, 1+\gamma)\}.$$

As x is the limit of LSOP solutions, S is the union of at most 4 intervals. Our previous observations and the special symmetry of x imply that for the Lebesgue measure $\mu(S_\gamma)$ of S_γ, we have estimation $\mu(S_\gamma) \leq 4 \cdot 4\gamma = 16\gamma$. Similarly, the measure of

$$S_\gamma^n = \{s \in [-1, 0] : \ x^n(s) \in (-1-\gamma, -1+\gamma) \cup (1-\gamma, 1+\gamma)\}$$

is not larger than 16γ for all sufficiently large n by Proposition 5.3 (iv).

We claim that for $t \in [0, 1]$,

$$\lim_{n\to\infty} \int_0^t e^{-(t-s)} f^{7,\varepsilon^n}(x^n(s-1))\, \mathrm{d}s = \int_0^t e^{-(t-s)} f^{7,0}(x(s-1))\, \mathrm{d}s,$$

that is to each $\eta > 0$ small, there corresponds $n_* \geq 1$ so that

$$\left| \int_0^t e^{-(t-s)} \left[f^{7,0}(x(s-1)) - f^{7,\varepsilon^n}(x^n(s-1)) \right] \mathrm{d}s \right| \leq$$

$$\int_0^t e^{-(t-s)} \left| f^{7,0}(x(s-1)) - f^{7,0}(x^n(s-1)) \right| \mathrm{d}s$$

$$+ \int_0^t e^{-(t-s)} \left| f^{7,0}(x^n(s-1)) - f^{7,\varepsilon^n}(x^n(s-1)) \right| \mathrm{d}s < \eta$$

for all $n \geq n_*$ and for all $t \in [0, 1]$. Set $\eta > 0$ and $0 < \gamma < \min\{\delta/2, \eta/224\}$. There exists $n_1 = n_1(\gamma) \geq 1$ so that for $n \geq n_1$, we have

$$f^{7,0}(x(s-1)) - f^{7,0}(x^n(s-1)) = 0 \quad \text{for } s-1 \notin S_\gamma,$$

and

$$\left| f^{7,0}(x(s-1)) - f^{7,0}(x^n(s-1)) \right| \leq 7 \quad \text{for } s-1 \in S_\gamma.$$

Therefore the first term is not larger than $7 \cdot 16\gamma t \leq 112\gamma < \eta/2$ for $n \geq n_1$. Also there is $n_2 = n_2(\gamma) \geq 1$ so that $\varepsilon^n < \gamma$ for all $n \geq n_2$. Then for $s - 1 \notin S_\gamma^n$,

$$f^{7,0}(x^n(-1+s)) - f^{7,\varepsilon^n}(x^n(-1+s)) = 0,$$

and for $s - 1 \in S_\gamma^n$,

$$\left| f^{7,0}(x^n(-1+s)) - f^{7,\varepsilon^n}(x^n(-1+s)) \right| \leq 7.$$

So the second term is is not larger than $7 \cdot 16\gamma t \leq 112\gamma < \eta/2$ if $n \geq n_2$. Set $n_* = \min\{n_1, n_2\}$. The claim is verified.

It follows that for all $t \in [0,1]$,

$$x(t) = \lim_{n\to\infty} x^n(t) = \lim_{n\to\infty} \left(e^{-t} x^n(0) + \int_0^t e^{-(t-s)} f^{7,\varepsilon^n}(x^n(s-1))\,\mathrm{d}s \right)$$
$$= e^{-t} x(0) + \int_0^t e^{-(t-s)} f^{7,0}(x(s-1))\,\mathrm{d}s,$$

that is, x satisfies Eq. (3.1) with $K=7$ and $\varepsilon = 0$ for all $t \in [0,1]$. It is analogous to show that x satisfies the equation on $[1,2]$. As $x_\omega = x_0$, we gain that x is a solution on $\mathbb{R}$.

Proposition 6.4 yields $x_0 = \Sigma(a^*,0)$ or $x_0 = \widetilde{\Sigma}(\widetilde{a},0)$. Suppose $x_0 = \Sigma(a^*,0)$ for example. Note that as x is of special symmetry, the construction of $\Sigma(a^*,0)$ yields that

$$a^* = \left(t_4 + 2 - \omega, t_1 - t_4 + \frac{\omega}{2}, t_4 - t_1 \right).$$

Proposition 5.3 gives that if n is large enough, then there exist uniquely defined $-1 < t_1^n < t_2^n < t_3^n < t_4^n < 0$ with

$$x^n(t_1^n) = 1, x^n(t_2^n) = 1 + \varepsilon^n, x^n(t_3^n) = 1 + \varepsilon^n, x^n(t_4^n) = 1.$$

Also, $\lim_{n\to\infty} t_1^n = \lim_{n\to\infty} t_2^n = t_1$ and $\lim_{n\to\infty} t_3^n = \lim_{n\to\infty} t_4^n = t_4$.

It follows from the definition of U_0^3, that $t_1 + 2 < \omega$ and $t_4 < t_1 + 1 - \omega/2$. Thus there exists $n_{**} \geq 1$ so that for $n \geq n_{**}$, we have $t_2^n + 2 < \omega^n$ and $t_4^n < t_1^n + 1 - \omega^n/2$. By Proposition 5.3 (v), $x_0^n = \Sigma(a^n, \varepsilon^n)$ for $n \geq n_{**}$, where

$$a^n = \left(t_3^n + 2 - \omega^n, t_1^n - t_4^n + \frac{\omega^n}{2}, t_3^n - t_2^n \right)$$

is a fixed point of $F(\cdot, \varepsilon^n)$. According to the proof of Proposition 3.11, there is a neighborhood N of a^* in $(0,1)^3$ so that the fixed point of $F(\cdot,\varepsilon)$ is unique in N for $\varepsilon \in [0, \varepsilon_0)$. As a^n is arbitrary close to a^*, we may suppose that $a^n \in N$ and thus $a^n = a^*(\varepsilon^n)$, a contradiction to our initial assumption.

At last suppose $x_0 = \widetilde{\Sigma}(\widetilde{a},0)$. Then with the aid of Proposition (5.3) (vi), one can verify the existence of $\widetilde{n} \geq 1$ so that $x_0^n = \widetilde{\Sigma}(\widetilde{a}(\varepsilon^n), \varepsilon^n)$ for $n \geq \widetilde{n}$, which is a contradiction again. □

Consider $K = 7$ and $\varepsilon \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$. Proposition 4.5 implies that there exists $\delta_0 = \delta_0(\varepsilon) > 0$ so that if $f \in C_b^1(\mathbb{R},\mathbb{R})$ with $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_0$, and (H1) holds for f, then Eq. (1.1) with $\mu = 1$ and nonlinearity f has two normalized LSOP solutions $p = p(f) : \mathbb{R} \to \mathbb{R}$ and $q = q(f) : \mathbb{R} \to \mathbb{R}$.

Proposition 6.7. *Set $\mu = 1$, $K = 7$. To each $\varepsilon \in (0, \varepsilon_*)$, where $\varepsilon_* \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ is given by Proposition 6.6, there corresponds $\delta_1 = \delta_1(\varepsilon) \in (0, \delta_0(\varepsilon))$ such that if*

$f \in C_b^1(\mathbb{R},\mathbb{R})$ satisfies (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1$, then Eq. (1.1) *admits at most two normalized LSOP solutions.*

Proof. Suppose for contradiction that a sequence $(f^n)_{n=0}^{\infty}$ exists in $C_b^1(\mathbb{R},\mathbb{R})$ with

$$\left\|f^n - f^{7,\varepsilon}\right\|_{C_b^1} < 1/n \text{ for } n \in \mathbb{N}$$

so that for $n \in \mathbb{N}$, f^n satisfies (H1), and the equation

$$\dot{x}(t) = -x(t) + f^n(x(t-1)) \tag{6.8}$$

has a normalized LSOP solution $x^n : \mathbb{R} \to \mathbb{R}$ with $x_0^n \notin \{p_0(f^n), q_0(f^n)\}$, where LSOP solutions $p(f^n)$ and $q(f^n)$ are given by Proposition 4.5. Note that $\|f^n - f^{7,\varepsilon}\|_{C_b^1} < \delta_0$ for all large n, hence it is no restriction to assume that $p(f^n)$ and $q(f^n)$ exist for all $n \geq 1$. Let $\omega^n \in (1,2)$ denote the minimal period of x^n, $n \in \mathbb{N}$. Since

$$\sup_{x\in\mathbb{R}} |f^n(x)| \leq \|f^n\|_{C_b^1} \leq \left\|f^{7,\varepsilon}\right\|_{C_b^1} + 1 < \infty, \quad n \in \mathbb{N},$$

Proposition 2.6 yields that $\|x_t^n\| \leq \|f^{7,\varepsilon}\|_{C_b^1} + 1$ and thus $\|\dot{x}_t^n\| \leq 2\|f^{7,\varepsilon}\|_{C_b^1} + 2$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Applying the Arzelà–Ascoli theorem, we may suppose that $\omega^n \to \omega \in [1,2]$ as $n \to \infty$, and x^n converges to a continuous function $x : \mathbb{R} \to \mathbb{R}$ uniformly on each compact subset of $\mathbb{R}$. Then it is easy to see that x is a solution of Eq. (3.1) with minimal period $\omega \in [1,2]$. Proposition 2.4 excludes the possibility that the period is 1, Proposition 2.4 and Proposition 5.2 exclude the possibility that the period is 2. So $\omega \in (1,2)$. As x is necessary of monotone type, this yields that $V(x_t) = V(x_0) = 2$ for all $t \in \mathbb{R}$. As x^n, $n \in \mathbb{N}$, is an LSOP solution, it is also easy to see that x is of large amplitude, i. e. $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. We conclude that x is an LSOP solution of (3.1). Hence Proposition 6.6 implies we may assume that x_0 is either $\Sigma(a^*(\varepsilon), \varepsilon)$ or $\widetilde{\Sigma}(\widetilde{a}(\varepsilon), \varepsilon)$. If n is chosen large enough, then f^n is arbitrarily close to $f^{7,\varepsilon}$ in C_b^1 norm, $x_0^n \in \{x_0\} + V$ and $\omega^n \in (\omega - \nu, \omega + \nu)$, where V and ν are given by Proposition 2.1. So Proposition 2.1 gives x^n equals $p(f^n)$ or $q(f^n)$, a contradiction to our initial assumption. □

The proof of Theorem 1.1. Fix $\mu = 1$, $K = 7$ and $\varepsilon \in (0, \varepsilon_*)$. Choose a nonlinearity $f \in C_b^1(\mathbb{R},\mathbb{R})$ satisfying (H1) so that $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1(\varepsilon) < \delta_0(\varepsilon)$. Then Theorem 1.1 follows from Propositions 4.5 and 6.7. □

7. Rapidly Oscillatory Periodic Solutions

We give conditions for the nonexistence of rapidly oscillatory solutions.

Proposition 7.1. *For $K \leq 8$ and $\varepsilon \in (0,1)$, Eq. (3.1) has no periodic solution $p : \mathbb{R} \to \mathbb{R}$ with $V(p_t) \geq 4$ for $t \in \mathbb{R}$.*

Proof. Propositions 5.1, 5.2 and 6.1 give that Eq. (3.1) has no periodic solution for $K \in (0,1]$. Set $K > 1$ and $\varepsilon \in (0,1)$. If $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution of Eq. (3.1), then it is of monotone type and of special symmetry. Hence if $V(p_t) \geq 4$, then $3\omega/2 < 1$, where $\omega > 0$ is the minimal period of p. Proposition 6.1 gives that

$$1 > \frac{3}{2}\omega \geq 6\ln\frac{K}{K-1} + 3\ln\frac{K+1}{K} > 9\ln\frac{K+1}{K},$$

that is

$$\begin{aligned}\frac{1}{K} < e^{\frac{1}{9}} - 1 &= \frac{1}{9} + \frac{\left(\frac{1}{9}\right)^2}{2!} + \frac{\left(\frac{1}{9}\right)^3}{3!} + ... \\ &< \frac{1}{9}\left(1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + ...\right) = \frac{1}{9}\frac{1}{1-\frac{1}{9}} = \frac{1}{8}.\end{aligned}$$

Thus $K > 8$ and the statement is verified. □

Proposition 7.2. *Set $\mu = 1$, $K = 7$. To each $\varepsilon \in (0,\varepsilon_*)$, where $\varepsilon_* \in (0, \min(\varepsilon_0, \widetilde{\varepsilon}_0))$ is given by Proposition 6.6, there corresponds $\delta_2 = \delta_2(\varepsilon) > 0$ such that if $f \in C_b^1(\mathbb{R},\mathbb{R})$ satisfies hypothesis (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_2$, then Eq. (1.1) with $\mu = 1$ and nonlinearity f has no periodic solutions oscillating rapidly around 0.*

Proof. Suppose for contradiction that there is a sequence $(f^n)_1^\infty$ in $C_b^1(\mathbb{R},\mathbb{R})$ with $\|f - f^{7,\varepsilon}\|_{C_b^1} \to 0$ as $n \to \infty$ so that for $n \in \mathbb{N}$, (H1) holds for f^n, and

$$\dot{x}(t) = -x(t) + f^n(x(t-1))$$

has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$, with $V(p_t^n) > 2$ for all $t \in \mathbb{R}$. Applying the Arzelà–Ascoli theorem, we get that there exists a continuous function $p : \mathbb{R} \to \mathbb{R}$ such that p^n, $\dot{p}^n$ converge to p, $\dot{p}$ uniformly on compact subsets of $\mathbb{R}$, respectively. Then p is a periodic solution of Eq. (3.1) with feedback function $f^{7,\varepsilon}$.

As $V(p_t^n) > 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, it is clear that $\max_{t\in\mathbb{R}} p^n(t) > 0$ for all $n \geq 1$. Applying the argument in the proof of Proposition 2.5 for periodic solution p^n, one obtains that $\max_{t\in\mathbb{R}} p^n(t) > \xi_1 > 1$ for all $n \geq 1$. Hence $\max_{t\in\mathbb{R}} p(t) > 0$. Using this estimate and the reasoning in the proof of Proposition 2.5 for the second time, $\max_{t\in\mathbb{R}} p(t) > \xi_1$ follows. Similarly, $\min_{t\in\mathbb{R}} p(t) < \xi_{-1}$.

As p is periodic, $V(p_t)$ is the same constant for all $t \in \mathbb{R}$. As p oscillates around 0, $V(p_t) \geq 2$ for all $t \in \mathbb{R}$. If $V(p_t) \equiv 2$, then p is an LSOP solution, and it is either $x^{\Sigma(a^*(\varepsilon),\varepsilon)}$ or $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon),\varepsilon)}$ up to time translation. By Proposition 5.3, the zeros of p are simple. As $p^n \to p$ and $\dot{p}^n \to \dot{p}$ uniformly on compact subsets of $\mathbb{R}$, we obtain that $V(p_t^n) \equiv 2$ for all large n, a contradiction to the choice of p^n. $V(p_t) > 4$ contradicts Proposition 7.1. The proof is complete. □

8. Connecting Orbits

This section assumes that we are in the situation of Theorem 1.1, namely $\mu = 1$, $f \in C^1$ satisfies (H1), furthermore $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are the normalized LSOP solutions of Eq. (1.1) with $p(\mathbb{R}) \subsetneq q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$.

Consider the C^1-smooth Poincaré return map P defined in a small neighborhood of p_0 in $H = \{\varphi \in C : \varphi(-1) = 0\}$ with fixed point p_0. Theorem 1.1 states that p_0 is hyperbolic and $DP(p_0)$ has exactly two eigenvalues $\lambda_1 > \lambda_2$ with absolute value greater than 1. Let H_s and H_u be the closed subspaces of H chosen so that $H = H_s \oplus H_u$, H_s and H_u are invariant under $L = DP(p_0)$, and the spectra σ_s and σ_u of the induced maps $H_s \ni x \mapsto Lx \in H_s$ and $H_u \ni x \mapsto Lu \in H_u$ are contained in $\{\mu \in \mathbb{C} : |\mu| < 1\}$ and $\{\mu \in \mathbb{C} : |\mu| > 1\}$, respectively. Then H_u is 2-dimensional (Appendix VII in [17]).

The unstable manifold. According to Appendix I in [17], there exist convex bounded neighborhoods N_s, N_u of 0 in H_s, H_u, respectively, and a C^1-map $w : N_u \to H_s$ with range in N_s so that $w(0) = 0$, $Dw(0) = 0$, and the subset

$$\mathcal{W}^u(p_0) = \{p_0 + x + w(x) : x \in N_u\}$$

of C is equal to

$$\big\{x \in p_0 + N_s + N_u : \text{there is a trajectory } (x_n)_{-\infty}^0 \text{ of } P \text{ in } p_0 + N_s + N_u \text{ with } x_0 = x \text{ and } x_n \to p_0 \text{ as } n \to -\infty\big\}.$$

$\mathcal{W}^u(p_0)$ is called the (2-dimensional) local unstable manifold of P at p_0.

The leading unstable manifold. Let H_u^1, H_u^2 be the linear subspaces in H_u generated by v_1, v_2, the eigenvectors corresponding to λ_1, λ_2, respectively. Then $H_u = H_u^1 \oplus H_u^2$. Set β so that $1 < \lambda_2 < \beta < \lambda_1$. There exist $\delta_0 > 0$ and a C^1-map $\tilde{w} : (-\delta_0, \delta_0) v_1 \to H_u^2 \oplus H_s$ with $\tilde{w}(0) = 0$ and $D\tilde{w}(0) = 0$ such that for $\delta^* \in (-\delta_0, \delta_0)$, there is a trajectory $(x_n)_{-\infty}^0$ of P with $x_0 = p_0 + \tilde{w}(\delta^* v_1) + \delta^* v_1$ and with $\beta^{-n}(x_n - p_0) \to 0$ as $n \to -\infty$. Moreover, x_n belongs to

$$\mathcal{W}_1^u(p_0) = \{p_0 + \tilde{w}(\delta v_1) + \delta v_1 : |\delta| < \delta_0\}$$

for $n \le 0$. Then $\mathcal{W}_1^u(p_0)$ is the leading unstable manifold of P at p_0. It is a 1-dimensional submanifold of $\mathcal{W}^u(p_0)$.

Similarly, there is a Poincaré map (also denoted by P) with fixed point q_0. By Theorem 1.1, $DP(q_0)$ has exactly one eigenvalue with absolute value greater than 1.

$\mathcal{W}^u(q_0)$ denotes the (1-dimensional) unstable manifold of P at q_0. The characterization of $\mathcal{W}^u(q_0)$ is analogous to the one given for $\mathcal{W}_1^u(p_0)$.

The unstable set of the orbit $\mathcal{O}_p = \{p_t : t \in \mathbb{R}\}$ is defined as

$$\mathcal{W}^u(\mathcal{O}_p) = \{x_0 : \ x : \mathbb{R} \to \mathbb{R} \text{ is a solution of } (1.1),\ \alpha(x) \text{ exists and } \alpha(x) = \mathcal{O}_p\}.$$

It is the forward extension of $\mathcal{W}^u(p_0)$:

$$\mathcal{W}^u(\mathcal{O}_p) = \{x_t^\varphi : \ \varphi \in \mathcal{W}^u(p_0),\ t \geq 0\}.$$

Set $\mathcal{W}^u(\mathcal{O}_q)$ is defined and described analogously. We also introduce the leading unstable set

$$\mathcal{W}_1^u(\mathcal{O}_p) = \{x_t^\varphi : \ \varphi \in \mathcal{W}_1^u(p_0),\ t \geq 0\}.$$

We say $\varphi \leq \psi$ for $\varphi, \psi \in C$ if $\varphi(s) \leq \psi(s)$ for all $s \in [-1, 0]$. Relation $\varphi < \psi$ holds if $\varphi \leq \psi$ and $\varphi \neq \psi$. In addition, $\varphi \ll \psi$ if $\varphi(s) < \psi(s)$ for all $s \in [-1, 0]$. Relations "$\geq$", "$>$" and "$\gg$" are defined analogously.

The semiflow Φ is monotone in the following sense.

Proposition 8.1. *Suppose $\varphi, \psi \in C$ with $\varphi \neq \psi$. Then $x_t^\varphi \neq x_t^\psi$ for all $t \geq 0$. If $\varphi < \psi$ ($\varphi > \psi$), then $x_t^\varphi \ll x_t^\psi$ $\left(x_t^\varphi \gg x_t^\psi\right)$ for all $t > 1$. In addition, if $\varphi \ll \psi$ ($\varphi \gg \psi$), then $x_t^\varphi \ll x_t^\psi$ $\left(x_t^\varphi \gg x_t^\psi\right)$ for all $t \geq 0$.*

The assertion follows easily from the variation-of-constant formula. For a proof we refer to [26].

Since $p(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$ and $q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$,

$$\mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q) \subset \mathcal{A}$$

by Proposition 8.1. Consequently, $\{x_t^\varphi : t \in \mathbb{R}\}$ is precompact for each $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$.

We need a few more propositions before proving Theorem 1.2.

Let p^0 denote the periodic solution of Eq. (3.1) with $K = 7$ and $\varepsilon = 0$ determined by the unique fixed point a^* of $F(\cdot, 0)$ in U_0^3.

Recall that if $\mu = 1$ and $K = 7$, then for each $\varepsilon \in (0, \varepsilon_*)$, there exists $\delta_1(\varepsilon) > 0$ such that if a nonlinearity $f \in C_b^1(\mathbb{R}, \mathbb{R})$ satisfies (H1), and $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1(\varepsilon)$, then the statement of Theorem 1.1 holds for f. Without loss of generality, we may assume that $\delta_1(\varepsilon) \to 0+$ as $\varepsilon \to 0+$. Hence we may assume that

$$\max_{-1 \leq t \leq 2} \left| p(t) - p^0(t) \right| \to 0 \quad \text{as } \varepsilon \to 0+. \tag{8.1}$$

We also have $\xi_1 \to 1$ and $\xi_2 \to 7$ as $\varepsilon \to 0+$.

Proposition 8.2. *Let $r:\mathbb{R}\to\mathbb{R}$ be a periodic solution of Eq. (1.1) either with range in $(0,\xi_2)$ and with $V\left(r_t-\hat{\xi}_1\right)=2$ for all $t\in\mathbb{R}$, or with range in $(\xi_{-2},0)$ and with $V\left(r_t-\hat{\xi}_{-1}\right)=2$ for all $t\in\mathbb{R}$. If $\varepsilon>0$ is sufficiently small, then $V(p_t-r_s)=2$ for all $t\in\mathbb{R}$ and $s\in\mathbb{R}$.*

Proof. We consider the case when r has range in $(0,\xi_2)$ and $V\left(r_t-\hat{\xi}_1\right)=2$ for all $t\in\mathbb{R}$. The other case is analogous.

By Proposition 2.4, $V(p_t-r_s)$ is the same constant for all $t\in\mathbb{R}$ and $s\in\mathbb{R}$. Thus it is sufficient to find a pair $(t,s)\in\mathbb{R}\times\mathbb{R}$ with $V(p_t-r_s)=2$. It is obvious that $V(p_t-r_s)>0$ for all $(t,s)\in\mathbb{R}\times\mathbb{R}$.

Let $\omega^0,\bar{\omega},\rho$ denote the minimal periods of p^0,p,r, respectively. Define $t_1,s_1,t_4,s_2,\tau=\omega^0-1$ for p^0 as in Section 3. Set $z=-1+\omega^0/2$. Then p^0 strictly increases on $[-1,s_1]$, decreases on $[s_1,z]$, $p^0(t)<0$ for $t\in(z,\tau)$, $p^0(-1)=p^0(z)=p^0(\tau)=0$ and $p^0(t_1)=p^0(t_4)=1$. As $f'(x)>0$ for all $x\in\mathbb{R}$, Theorem 7.1 in [22] implies that p is strictly monotone between two extremum points. So there exist $\bar{t}_1,\bar{s}_1,\bar{t}_4,\bar{z},\bar{\tau}=\bar{\omega}-1$ such that p strictly increases on $[-1,\bar{s}_1]$, decreases on $[\bar{s}_1,\bar{z}]$, $p(t)<0$ for $t\in(\bar{z},\bar{\tau})$, $p(-1)=p(\bar{z})=p(\bar{\tau})=0$ and $p(\bar{t}_1)=p(\bar{t}_4)=\xi_1$ (see Fig. 11). Property (8.1) implies that $\bar{t}_1\to t_1$, $\bar{s}_1\to s_1$, $\bar{t}_4\to t_4$, $\bar{z}\to z$, $\bar{\tau}\to\tau$ as $\varepsilon\to 0+$.

From Section 3 we know that

$$\tau-1>t_1,\ 1>\tau-t_4>\frac{\omega^0}{2}>\frac{1}{2}\text{ and }\omega^0>1.$$

Fix δ_0 with

$$\delta_0\in\left(0,\min\left\{s_1-t_1,z-t_4,t_4-s_1,\tau-1-t_1,\omega^0-1\right\}\right).$$

Choose ε_0 so small that for each $\varepsilon\in(0,\varepsilon_0)$, $\bar{\tau}-1>\bar{t}_1$, $1>\bar{\tau}-\bar{t}_4>1/2$, $\bar{\omega}>1$ and also

$$\delta_0\in\left(0,\min\left\{\bar{s}_1-\bar{t}_1,\bar{z}-\bar{t}_4,\bar{t}_4-\bar{s}_1,\bar{\tau}-1-\bar{t}_1,\bar{\omega}-1\right\}\right).$$

hold.

Claim. There exists $\varepsilon_1\in(0,\varepsilon_0)$ such that for all $\varepsilon\in(0,\varepsilon_1)$,

(i) if $r(t_0+\sigma)=\xi_1$ for some $t_0\in[\bar{t}_1+\delta_0,\bar{s}_1]$ and $\sigma\in\mathbb{R}$, then $r(\sigma+s)<p(s)$ for all $s\in[t_0,\bar{s}_1]$ (see Fig. 10),

(ii) if $r(t_0+\sigma)=\xi_1$ for some $t_0\in[\bar{t}_4+\delta_0,\bar{z}]$ and $\sigma\in\mathbb{R}$, then $r(\sigma+s)>p(s)$ for all $s\in[t_0,\bar{z}]$,

(iii) if $r(t_0+\sigma)=\xi_1$ for some $t_0\in[\bar{s}_1,\bar{t}_4-\delta_0]$ and $\sigma\in\mathbb{R}$, then $r(\sigma+s)<p(s)$ for all $s\in[\bar{s}_1,t_0]$.

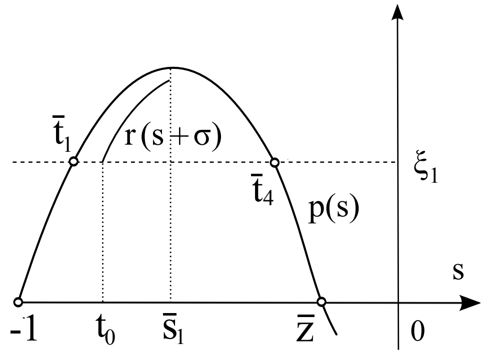

FIGURE 10. The periodic solutions in Claim (i)

Proof of (i). For r, the differential inequality

$$\dot{r}(t) \leq -r(t) + f(\xi_2)$$

holds for all $t \in \mathbb{R}$. Hence

$$r(\sigma + s) \leq \xi_1 e^{t_0 - s} + \left(1 - e^{t_0 - s}\right) f(\xi_2) \text{ for } s \geq t_0.$$

For a fixed $t_0 \in [t_1 + \delta_0, s_1]$, the right side of the inequality tends to $7 - 6e^{t_0 - s}$ as $\varepsilon \to 0+$ uniformly in $s \in [t_0, s_1]$. Using $p^0(s) = 7 - 6e^{t_1 - s}$, $s \in [t_1, s_1]$, one finds

$$\min_{s \in [t_0, s_1]} \left(p^0(s) - \left(7 - 6e^{t_0 - s}\right)\right) = 6\left(1 - e^{t_1 - t_0}\right) \min_{s \in [t_0, s_1]} e^{t_0 - s} \geq 6\left(e^{\delta_0} - 1\right) e^{t_1 - s_1} > 0.$$

These facts and (8.1) imply Claim (i) for all sufficiently small $\varepsilon > 0$.

Assertions (ii) and (iii) of the Claim can be shown analogously, therefore we omit the details.

Let u_i, $i \in \{0, 1, 2, 3, 4\}$, be given so that $u_0 < u_1 < u_2 < u_3 < u_4$, $u_4 = u_0 + \rho$, $r(u_0) = r(u_2) = \xi_1$, $r(u_1) = \min_{t \in \mathbb{R}} r(t) > 0$ and $r(u_3) = \max_{t \in \mathbb{R}} r(t) < \xi_2$. Proposition 2.5 and Theorem 7.1 in [22] guarantee the existence of $u_0, .., u_4$ and the fact that r strictly increases on $[u_1, u_3]$ and strictly decreases on $[u_3, u_5]$ with $u_5 = u_1 + \rho$.

We distinguish two cases according to the length of $[u_2, u_4]$.

Case 1: $u_4 - u_2 \geq \bar{\tau} - \bar{t}_4$. As $\bar{\tau} - \bar{t}_4 > 1/2$ by the choice of ε, $u_4 - u_3 < \bar{\tau} - \bar{t}_4$ or $u_3 - u_2 < \bar{\tau} - \bar{t}_4$ holds because $u_4 - u_2 < 1$. So two subcases need to be considered.

Case 1.1: If $u_4 - u_3 < \bar{\tau} - \bar{t}_4$, set $y(t) = r(t - \bar{\tau} + u_4)$. Then y decreases on $[\bar{\tau} - u_4 + u_3, \bar{\tau}]$, increases on $[\bar{\tau} - u_4 + u_1, \bar{\tau} - u_4 + u_3]$, decreases on $[\bar{\tau} - u_4 + u_0, \bar{\tau} - u_4 + u_1]$ and

$$y(\bar{\tau}) = y(\bar{\tau} - u_4 + u_2) = y(\bar{\tau} - u_4 + u_0) = \xi_1.$$

Case 1.1.1: If in addition, $\bar{\tau} - u_4 + u_2 \in [\bar{s}_1, \bar{t}_4]$, then $p - y$ has one sign change on $[\bar{\tau} - 1, \bar{\tau}]$ since $y(t) < \xi_1$ for $t \in (\bar{\tau} - u_4 + u_0, \bar{\tau} - u_4 + u_2)$, $p(t) > \xi_1$ for $t \in$

$(\bar{\tau}-1, \bar{\tau}-u_4+u_2)$, $\bar{\tau}-u_4+u_0<\bar{\tau}-1$, p decreases on $[\bar{\tau}-u_4+u_2, \bar{t}_4]$, y increases on $[\bar{\tau}-u_4+u_2, \bar{t}_4]$, and $y(t)>\xi_1>p(t)$ for $t \in (\bar{t}_4, \bar{\tau})$ (see Fig. 11).

Case 1.1.2: If $\bar{\tau}-u_4+u_2<\bar{s}_1$, then $\bar{\tau}-u_4+u_2 \in (\bar{\tau}-1, \bar{s}_1)$. The choice of ε implies $\bar{t}_1+\delta_0<\bar{\tau}-1$, hence $\bar{\tau}-u_4+u_2 \in (\bar{t}_1+\delta_0, \bar{s}_1)$, and Claim (i) can be applied to get $y(t)<p(t)$ for all $t \in [\bar{\tau}-u_4+u_2, \bar{s}_1]$. As $\bar{\tau}-u_4+u_0<\bar{\tau}-1$, inequality $y(t)<\xi_1$ holds for all $t \in [\bar{\tau}-1, \bar{\tau}-u_4+u_2)$. Now it is obvious that $p-y$ has exactly one sign change on $[\bar{\tau}-1, \bar{\tau}]$ (see Fig. 12).

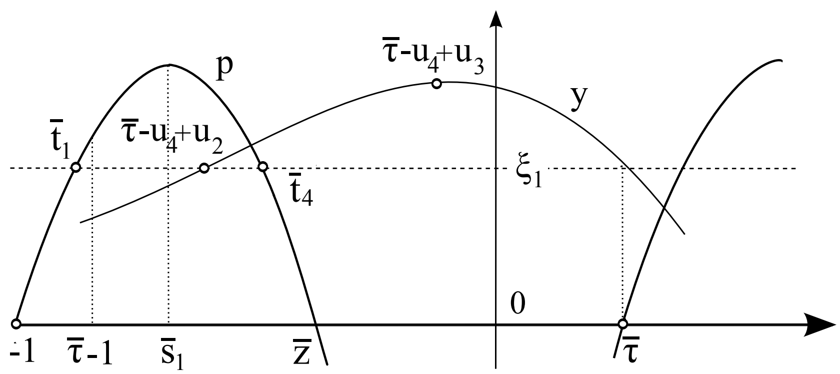

FIGURE 11. Case 1.1.1 in the proof of Proposition 8.2

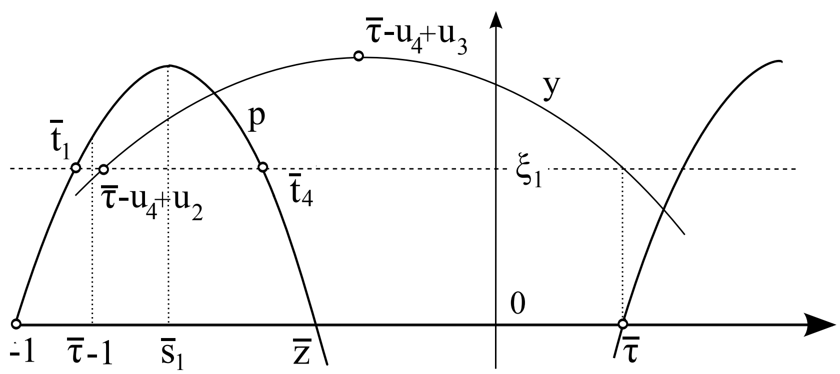

FIGURE 12. Case 1.1.2 in the proof of Proposition 8.2

Case 1.2: If $u_3-u_2<\bar{\tau}-\bar{t}_4$, define $y(t)=r(t-\bar{t}_4+u_2)$. Then

$$y(\bar{t}_4+u_0-u_2)=y(\bar{t}_4)=y(\bar{t}_4+u_4-u_2)=\xi_1,$$

y decreases on $[\bar{t}_4+u_0-u_2, \bar{t}_4+u_1-u_2]$, increases on $[\bar{t}_4+u_1-u_2, \bar{t}_4+u_3-u_2]$ and decreases on $[\bar{t}_4+u_3-u_2, \bar{t}_4+u_5-u_2]$. With this choice of y, function $p-y$ has exactly one sign change on $[\max\{\bar{\tau}-1, \bar{t}_4+u_0-u_2\}, \bar{\tau}]$, since

$$y(t)<\xi_1<p(t) \quad \text{for } t \in (\max\{\bar{\tau}-1, \bar{t}_4+u_0-u_2\}, \bar{t}_4)$$

and

$$p(t) < \xi_1 < y(t) \quad \text{for } t \in (\bar{t}_4, \bar{\tau}).$$

The proof is clearly complete if $\bar{\tau} - 1 \geq \bar{t}_4 + u_0 - u_2$, hence we may suppose that $\bar{\tau} - 1 < \bar{t}_4 + u_0 - u_2$.

Case 1.2.1: If in addition, $\bar{t}_4 + u_4 - u_2 \in [\bar{\tau}, \bar{s}_1 + \bar{\omega}]$, then $p - y$ has at most one sign change on $[\bar{\tau}, \bar{t}_4 + u_4 - u_2]$, since y decreases and p increases on this interval. Therefore $p - y$ has at most two sign changes on $[\bar{t}_4 + u_0 - u_2, \bar{t}_4 + u_4 - u_2]$, which interval has length $\rho > 1$.

Case 1.2.2: If $\bar{t}_4 + u_4 - u_2 > \bar{s}_1 + \bar{\omega}$, then $\bar{t}_4 + u_4 - u_2 \in [\bar{s}_1 + \bar{\omega}, \bar{t}_4 + \bar{\omega} - \delta_0]$, as $u_4 - u_2 < 1 < \bar{\omega} - \delta_0$. According to Claim (iii), $y(t) < p(t)$ for $t \in [\bar{s}_1 + \bar{\omega}, \bar{t}_4 + u_4 - u_2]$. As p increases and y increases on $[\bar{\tau}, \bar{s}_1 + \bar{\omega}]$, function $p - y$ has at most one sign change on this interval. So $p - y$ has at most two sign changes on $[\bar{t}_4 + u_0 - u_2, \bar{t}_4 + u_4 - u_2]$.

Case 2: If $u_4 - u_2 < \bar{\tau} - \bar{t}_4$, then the proof is separated into three subcases according to the length of $[u_1, u_4]$.

Case 2.1: If $\bar{\tau} - \bar{t}_4 \leq u_4 - u_1$, set $y(t) = r(t - \bar{\tau} + u_4)$. As in the previous cases, the property that p and y are monotone on the intervals on which they are not bounded away from each other implies that $p - y$ has at most two sign changes on $[\bar{\tau} - 1, \bar{\tau}]$.

Case 2.2: If $\bar{z} - \bar{t}_4 \leq u_4 - u_1 < \bar{\tau} - \bar{t}_4$, choose $y(t) = r(t - \bar{t}_4 + u_1)$. Then $p - y$ has at most two sing changes on $[\bar{t}_4, \bar{t}_4 + 1]$.

Case 2.3: If $u_4 - u_1 < \bar{z} - \bar{t}_4$, set $y(t) = r(t - \bar{t}_4 + u_1)$ again. Note that as $u_1 - u_0 < 1 < \bar{\omega} - \delta_0$, inequality $\bar{t}_4 - u_1 + u_0 > \bar{t}_4 - \bar{\omega} + \delta_0$ holds. So either $\bar{t}_4 - \bar{\omega} + \delta_0 < \bar{t}_4 - u_1 + u_0 \leq \bar{z} - \bar{\omega}$ or $\bar{t}_4 - u_1 + u_0 > \bar{z} - \bar{\omega}$.

Case 2.3.1: If $\bar{t}_4 - u_1 + u_0 > \bar{z} - \bar{\omega}$, then $p - y$ admits at most two sign changes on $[\bar{t}_4 - u_1 + u_0, \bar{t}_4 - u_1 + u_4]$, which interval has length $\rho > 1$.

Case 2.3.2: If $\bar{t}_4 - \bar{\omega} + \delta_0 < \bar{t}_4 - u_1 + u_0 \leq \bar{z} - \bar{\omega}$, apply Claim (ii) to get that $y(t) > p(t)$ for all $t \in [\bar{t}_4 - u_1 + u_0, \bar{z} - \bar{\omega}]$. It follows that $p - y$ has at most two sign changes on $[\bar{t}_4 - u_1 + u_0, \bar{t}_4 - u_1 + u_4]$.

The proof is complete. □

Proposition 8.3. *Assume $x : \mathbb{R} \to \mathbb{R}$ is a solution of Eq.* (1.1) *with initial function $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$ such that x oscillates around $\xi \in \{\xi_{-1}, 0, \xi_1\}$. Then $V\left(x_t - \hat{\xi}\right) = 2$ for all $t \in \mathbb{R}$, where $\hat{\xi} \in C$ is the equilibrium $\hat{\xi}(s) = \xi$, $s \in [-1, 0]$. In addition, $V(x_{t+u} - p_t) = 2$ for all $u, t \in \mathbb{R}$ and $V(x_{t+u} - x_t) = 2$ for all $u \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$. If there exists $i \in \{-1, 1\}$ so that x oscillates around ξ_i, then $V(x_{t+u} - x_t^i) = 2$ for all $u, t \in \mathbb{R}$, where $x^i : \mathbb{R} \to \mathbb{R}$ is given by Proposition 2.7.*

Proof. Let x be a solution of Eq. (1.1) oscillating around $\xi \in \{\xi_{-1}, 0, \xi_1\}$ with $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$. Clearly, $x_0 \neq \hat{\xi}$, hence $x_t \neq \hat{\xi}$ for $t \in \mathbb{R}$ by Proposition 8.1.

Since $x_0 \in \mathcal{W}^u(p_0)$, there exists $(t_n)_0^\infty \subset \mathbb{R}$ so that $t_n \to -\infty$ as $n \to \infty$, $x_{t_n} \in \mathcal{W}^u(p_0)$ for $n \geq 0$ and $x_{t_n} \to p_0$ in C as $n \to \infty$. Clearly, $p_0 \in \mathcal{A}$ and $x_t \in \mathcal{A}$ for all $t \in \mathbb{R}$. The norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent on $\mathcal{A}$. Thus $x_{t_n} \to p_0$ also in C^1-norm as $n \to \infty$.

Let $\omega \in (1, 2)$ be the minimal period of p. Clearly, $V\left(p_t - \hat{\xi}\right) = 2$ for all $t \in [0, \omega)$, hence Lemma 2.3 (iii) gives that $p_t - \hat{\xi} \in R$ for all $t \in [0, \omega)$, where function class R is defined in Section 2. Lemma 2.2 implies that

$$2 = V\left(p_0 - \hat{\xi}\right) = \lim_{n\to\infty} V\left(x_{t_n} - \hat{\xi}\right).$$

Hence by Lemma 2.3 (i), $V\left(x_t - \hat{\xi}\right) \leq 2$ for all real t. If $V\left(x_{t^*} - \hat{\xi}\right) = 0$ for some $t^* \in \mathbb{R}$, that is $x_{t^*} < \hat{\xi}$ or $x_{t^*} > \hat{\xi}$, then Proposition 8.1 implies $x_t \ll \hat{\xi}$ or $x_t \gg \hat{\xi}$ for all $t > t^* + 1$, respectively. This is a contradiction as x oscillates around ξ. So $V\left(x_t - \hat{\xi}\right) = 2$ for all $t \in \mathbb{R}$.

It is easy to deduce from the monotone property of p that $V(p_{t+\tau} - p_{t+\sigma}) = 2$ in case $t \in \mathbb{R}$, $\tau, \sigma \in [0, \omega)$ and $\sigma \neq \tau$. In consequence $p_{t+\tau} - p_{t+\sigma} \in R$ all for $t \in \mathbb{R}$ and $\sigma \neq \tau$.

Now choose any $u \in \mathbb{R}$. Using the continuity of the flow $\Phi_{\mathcal{A}}$, we obtain that $x_{t_n+u} \to p_u$ in C^1-norm as $n \to \infty$. By compactness, we may assume the existence of $\sigma \in [0, \omega)$ such that $p_{t_n} \to p_\sigma$ in C^1-norm as $n \to \infty$. If $\sigma \neq u$, then Lemma 2.2 implies that

$$2 = V(p_u - p_\sigma) = \lim_{n\to\infty} V(x_{t_n+u} - p_{t_n}),$$

and Lemma 2.3 (i) gives that $V(x_{t+u} - p_t) \leq 2$ for all real t. In case $\sigma = u$, observe that $x_{t_n+u+\varepsilon} \to p_{u+\varepsilon} \neq p_\sigma$ for any small $\varepsilon > 0$, thus we may use our previous result and Lemma 2.2 to get

$$V(x_{t+u} - p_t) \leq \liminf_{\varepsilon\to 0+} V(x_{t+u+\varepsilon} - p_t) \leq 2$$

for all real t.

Now assume that $V(x_{t^*+u} - p_{t^*}) = 0$ for some $t^* \geq 0$, that is $x_{t^*+u} \leq p_{t^*}$ or $x_{t^*+u} \geq p_{t^*}$. Suppose $x_{t^*+u} \leq p_{t^*}$ for example. As $x_0 \notin \mathcal{O}_p$, Proposition 8.1 gives $x_{t^*+u} \neq p_{t^*}$ and thus $x_{t^*+u+2} \ll p_{t^*+2}$. By [26], the set of those functions φ for which x_t^φ converges to an equilibrium as $t \to \infty$ is dense in C. Consequently there exists $\eta \in C$ so that x_t^η tends to one of the equilibrium points as $t \to \infty$, and $x_{t^*+u+2} \ll \eta \ll p_{t^*+2}$. As $x_{t+t^*+u+2} \ll x_t^\eta \ll p_{t+t^*+2}$ for all $t \geq 0$ again by Proposition 8.1, this equilibrium point is necessarily $\hat{\xi}_{-2}$ contradicting to the fact that x oscillates around ξ. One comes to the same conclusion assuming that $x_{t^*+u} \geq p_{t^*}$.

The argument confirming the rest of the claim is similar, so we leave it to the reader. To prove the last assertion, use Proposition 8.2. □

A second key tool besides the Lyapunov functional is the linear map $\pi : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$ introduced in Section 2. From the paper [22] of Mallet-Paret and Sell we know that π maps nontrivial periodic orbits of Eq. (1.1) into simple closed curves in $\mathbb{R}^2$, and the images of different periodic orbits are disjoint curves in $\mathbb{R}^2$. So

$$O_p : \mathbb{R} \ni t \mapsto \pi p_t \in \mathbb{R}^2, \; O_q : \mathbb{R} \ni t \mapsto \pi q_t \in \mathbb{R}^2,$$

$$O_1 : \mathbb{R} \ni t \mapsto \pi x_t^1 \in \mathbb{R}^2 \text{ and } O_{-1} : \mathbb{R} \ni t \mapsto \pi x_t^{-1} \in \mathbb{R}^2$$

are simple closed curves and disjoint. Here x^1 and x^{-1} are the periodic solutions given by Proposition 2.7. As $p(\mathbb{R}) \subsetneq q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$, $O_q \subset \text{ext}(O_p)$ and $\pi\hat{\xi}_{-2}$, $\pi\hat{\xi}_2$ belong to $\text{ext}(O_q)$. Also, $\pi\hat{0}, O_1, O_{-1} \in \text{int}(O_p)$. For the images of the unstable equilibria, we have $\pi\hat{\xi}_{-1} \in \text{int}(O_{-1})$ and $\pi\hat{\xi}_1 \in \text{int}(O_1)$. If $x : \mathbb{R} \to \mathbb{R}$ is periodic solution oscillating slowly around ξ_{-1} with $x(\mathbb{R}) \subset (\xi_{-2}, 0)$, then either $\{\pi x_t : t \in \mathbb{R}\} = O_{-1}$ or $\{\pi x_t : t \in \mathbb{R}\} \subset \text{int}(O_{-1})$ by Proposition 2.7. Similarly, for a periodic solution x oscillating slowly around ξ_1 with range in $(0, \xi_2)$, either $\{\pi x_t : t \in \mathbb{R}\} = O_1$ or $\{\pi x_t : t \in \mathbb{R}\} \subset \text{int}(O_1)$.

Note that as $p(-1) = q(-1) = 0$, $p(0) < 0$, $q(0) < 0$ and $O_q \subset \text{ext}(O_p)$, we have $q(0) < p(0) < 0$.

Corollary 8.4. *Let $x : \mathbb{R} \to \mathbb{R}$ be a solution of Eq.* (1.1) *with initial data $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$ such that x oscillates around $\xi \in \{\xi_{-1}, 0, \xi_1\}$. Then curve $S : \mathbb{R} \ni t \mapsto \pi x_t \in \mathbb{R}^2$ is simple and does not intersect O_p.*

Proof. Proposition 8.3 yields that $t \mapsto V(x_{t+u} - x_t)$ is finite and constant for all $u \in \mathbb{R} \setminus \{0\}$. If there exist $t \in \mathbb{R}$ and $u \in \mathbb{R} \setminus \{0\}$ such that $\pi x_t = \pi x_{t+u}$, then by Lemma 2.3 (ii), $V(x_{t+u} - x_t) < V(x_{t+u-2} - x_{t-2})$, a contradiction. So S is simple. It follows from Proposition 8.3 and Lemma 2.3 (ii) in a similar way that S and O_p are disjoint. □

The Proof of Theorem 1.2. Set $\mu = 1$, $K = 7$ and $\varepsilon \in (0, \varepsilon_*)$, where ε_* is given by Proposition 6.6. Choose nonlinearity $f \in C_b^1(\mathbb{R}, \mathbb{R})$ satisfying hypothesis (H1) so that $\|f - f^{7,\varepsilon}\|_{C_b^1} < \min\{\delta_1, \delta_2\}$. Then the conditions of Propositions 4.5, 6.7 and 7.2 are satisfied by f, which means that the statement of Theorem 1.1 holds, and Eq. (1.1) admits no rapidly oscillatory solutions.

Remark 8.5. We may assume that f satisfies hypothesis (H2) introduced in Section 2. As f is close to $f^{7,\varepsilon}$ in C_b^1-norm, it suffices to verify this statement for $f^{7,\varepsilon}$ with

$\varepsilon \in (0, \varepsilon_*)$. Recall that $f^{7,\varepsilon}$ is defined by

$$f^{7,\varepsilon}(x) = 7\rho\left(\frac{|x|-1}{\varepsilon}\right)\operatorname{sgn}(x)$$

for all $\varepsilon \in [0,1)$, where $\rho \in C^\infty$, $\rho(t) = 0$ for $t \leq 0$, $\rho(t) = 1$ for $t \geq 1$ and $\rho'(t) > 0$ for all $t \in (0,1)$. Set interval $I_\varepsilon = \rho^{-1}[1/7, (1+\varepsilon)/7]$. Clearly,

$$\min_{t \in I_\varepsilon} \rho'(t) \geq \min_{t \in \rho^{-1}\left[\frac{1}{7}, \frac{2}{7}\right]} \rho'(t) = m > 0.$$

As (H1) holds and $\xi_1 \in (1, 1+\varepsilon)$, there exists $t_0 \in I_\varepsilon$ such that $t_0 = (\xi_1 - 1)/\varepsilon$ and $\rho(t_0) = \xi_1/7$. We obtain that

$$\left(f^{7,\varepsilon}\right)'(\xi_1) = \frac{7}{\varepsilon}\rho'(t_0) \geq \frac{7m}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0+.$$

Similarly, $\left(f^{7,\varepsilon}\right)'(\xi_{-1}) \to \infty$ as $\varepsilon \to 0+$. So we may assume that $\varepsilon_* > 0$ is chosen so small that (H2) holds for $f^{7,\varepsilon}$ with $\varepsilon \in (0, \varepsilon_*)$.

Theorem 1.2 is a consequence of Claims 8.6–8.12 below.

Claim 8.6. $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}) = \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$.

Proof. Clearly, $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}) \supseteq \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$. Suppose $x : \mathbb{R} \to \mathbb{R}$ is a solution of (1.1) with $x_0 \in \mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$. Then $\alpha(x)$ contains no stable equilibrium point, as in this case x_0 would be the stable equilibrium itself. If $\hat{\xi}_1 \in \alpha(x)$, then Proposition 8.1 implies $x_t \in C_{0,2}$ for all $t \in \mathbb{R}$, a contradiction to $x_0 \notin \mathcal{A}_{0,2}$. Similarly, $\hat{\xi}_{-1} \notin \alpha(x)$. As x is necessarily bounded, the Poincaré–Bendixson theorem implies $\alpha(x)$ is a periodic orbit. Theorem 7.2 gives that there are no periodic orbits in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ besides $\mathcal{O}_p$ and $\mathcal{O}_q$. So $x_0 \in \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$. □

Claim 8.7. There exist connecting orbits from $\mathcal{O}_p$ and $\mathcal{O}_q$ to the equilibrium points $\hat{\xi}_{-2}$ and $\hat{\xi}_2$. Moreover, for each $\varphi \in \mathcal{W}_1^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ and for each $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$, $\omega(\varphi)$ is either $\left\{\hat{\xi}_{-2}\right\}$ or $\left\{\hat{\xi}_2\right\}$.

Proof. First consider the 1-dimensional leading unstable manifold $\mathcal{W}_1^u(p_0)$. By Appendix VII in [17], the eigenfunction v_1 corresponding to the greatest positive eigenvalue λ_1 of $DP(p_0)$ is strictly positive. Choose δ_1 so small that $\|D\tilde{w}(\delta v_1)\| < 1/2$ for $|\delta| < \delta_1$, where $\tilde{w}$ is the map introduced on page 64. Observe that

$$\tilde{w}(\delta v_1) + \delta v_1 = \int_0^1 D\tilde{w}(s\delta v_1)\,\delta v_1 \mathrm{d}s + \delta v_1 \gg 0$$

if $\delta \in (0, \delta_1)$, and $\tilde{w}(\delta v_1) + \delta v_1 \ll 0$ if $\delta \in (-\delta_1, 0)$. Setting

$$\eta_1 = p_0 + \frac{\delta_1}{2} v_1 + \tilde{w}\left(\frac{\delta_1}{2} v_1\right) \text{ and } \eta_2 = p_0 - \frac{\delta_1}{2} v_1 + \tilde{w}\left(-\frac{\delta_1}{2} v_1\right),$$

we get $\eta_1, \eta_2 \in \mathcal{W}_1^u(p_0)$ and $\eta_2 \ll p_0 \ll \eta_1$. By [26], there exist $\eta_1^+, \eta_1^-, \eta_2^+, \eta_2^- \in C$ such that

$$\eta_2^- \ll \eta_2 \ll \eta_2^+ \ll p_0 \ll \eta_1^- \ll \eta_1 \ll \eta_1^+,$$

and for $i = 1, 2$, $x_t^{\eta_i^-}$ and $x_t^{\eta_i^+}$ converge to one of the equilibrium points as $t \to \infty$. Since $\max_{t\in\mathbb{R}} p(t) > \xi_1$, $\min_{t\in\mathbb{R}} p(t) < \xi_{-1}$ and

$$x_t^{\eta_2^-} \ll x_t^{\eta_2^+} \ll p_t \ll x_t^{\eta_1^-} \ll x_t^{\eta_1^+} \quad \text{for all } t \geq 0$$

by Proposition 8.1, we obtain that

$$x_t^{\eta_2^-} \to \hat{\xi}_{-2},\ x_t^{\eta_2^+} \to \hat{\xi}_{-2},\ x_t^{\eta_1^-} \to \hat{\xi}_2 \text{ and } x_t^{\eta_1^+} \to \hat{\xi}_2 \text{ as } t \to \infty.$$

Using again Proposition 8.1, we get $x_t^{\eta_2} \to \hat{\xi}_{-2}$ and $x_t^{\eta_1} \to \hat{\xi}_2$ as $t \to \infty$.

For each $\varphi \in \mathcal{W}_1^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, there is a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) and a sequence $(t_n)_0^\infty$ such that $x_0 = \varphi$, $x_{t_n} \in \mathcal{W}_1^u(p_0) \setminus p_0$ for all $n \geq 0$ and $x_{t_n} \to p_0$ as $n \to \infty$. Hence there exist $\delta \in (-\delta_1, 0) \cup (0, \delta_1)$ and $n^* \geq 0$ so that $x_{t_{n^*}} = p_0 + \tilde{w}(\delta v_1) + \delta v_1$. The above reasoning shows that if $\delta < 0$, then $\omega(\varphi) = \left\{\hat{\xi}_{-2}\right\}$, and if $\delta > 0$, then $\omega(\varphi) = \left\{\hat{\xi}_2\right\}$.

Since $\mathcal{W}^u(q_0)$ is a 1-dimensional unstable manifold as well, and $\mathcal{W}^u(\mathcal{O}_q)$ is the forward extension of $\mathcal{W}^u(q_0)$, it is analogous to show that for each $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$, $\omega(\varphi)$ is either $\left\{\hat{\xi}_{-2}\right\}$ or $\left\{\hat{\xi}_2\right\}$, moreover these connections indeed exist. □

It remains to describe $\mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{W}_1^u(\mathcal{O}_p)$.

Claim 8.8. Suppose that for $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, the limit set $\omega(\varphi)$ is a non-constant periodic orbit. Then if the solution $x^\varphi : \mathbb{R} \to \mathbb{R}$ oscillates around 0, then $\omega(\varphi) = \mathcal{O}_q$. Otherwise $\omega(\varphi)$ is either $\mathcal{O}_{-1}$ or $\mathcal{O}_1$.

Proof. Suppose $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, and $\omega(\varphi)$ is a non-constant periodic orbit $\{r_t : t \in \mathbb{R}\}$.

First let us examine the case when $x^\varphi : \mathbb{R} \to \mathbb{R}$ oscillates around 0. Then as $\mathcal{W}^u(\mathcal{O}_p)$ is the forward extension of $\mathcal{W}^u(p_0)$, Proposition 8.3 implies $V(x_t^\varphi) = 2$ for all $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ fixed, there exists $(t_n)_0^\infty$ with $t_n \to \infty$ as $n \to \infty$ so that r_t is the limit of $x_{t_n}^\varphi$ in C. As we have seen before, this implies convergence also in C^1-norm. As the segments of any periodic solution belong to R, Lemma 2.2 gives $V(r_t) = \lim_{n\to\infty} V\left(x_{t_n}^\varphi\right) = 2$. In addition, Proposition 2.5 yields $r(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. Therefore r equals p or q apart from shift by Theorem 1.1. We claim that $\omega(\varphi) \neq \mathcal{O}_p$. Indeed, Corollary 8.4 implies $\mathbb{R} \ni t \mapsto \pi x_t^\varphi \in \mathbb{R}^2$ is a simple curve winding around $(0,0)$. This fact and $\operatorname{dist}(\pi x_t^\varphi, \pi\mathcal{O}_p) \to 0$

as $t \to \pm\infty$ give a contradiction by the Jordan curve theorem. So we obtain that if $x^{\varphi} : \mathbb{R} \to \mathbb{R}$ oscillates around 0, then $\omega(\varphi) = \mathcal{O}_q$.

Now assume x^{φ} is not oscillatory around 0. Then there exists $t_* \in \mathbb{R}$ such that $x^{\varphi}_{t_*} > 0$ or $x^{\varphi}_{t_*} < 0$. Suppose $x^{\varphi}_{t_*} > 0$ for example. Then $x^{\varphi}_t \gg 0$ for all $t > t_* + 1$. Necessarily $r(t) > 0$ for all $t \in \mathbb{R}$. Proposition 2.5 gives that

$$0 < \min_{t \in \mathbb{R}} r(t) < \xi_1 < \max_{t \in \mathbb{R}} r(t) < \xi_2.$$

As $\omega(\varphi) = \{r_t : t \in \mathbb{R}\}$, it follows that x^{φ} is also oscillatory around ξ_1. Therefore $V\left(x^{\varphi}_t - \hat{\xi}_1\right) = 2$ for all $t \in \mathbb{R}$ by Proposition 8.3. For each $t \in \mathbb{R}$, there corresponds a sequence $(t_n)_0^{\infty} \subset \mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ such that $x^{\varphi}_{t_n} \to r_t$ in C (and thus in C^1) as $n \to \infty$. Hence $V\left(r_t - \hat{\xi}_1\right) = 2$ for all $t \in \mathbb{R}$ by Lemma 2.2. We obtain that r is slowly oscillatory around ξ_1 and has range in $(0, \xi_2)$. Recall from Proposition 2.7 that the periodic solution $x^1 : \mathbb{R} \to \mathbb{R}$ is set so that it oscillates slowly around ξ_1 with $x^1(\mathbb{R}) \subset (0, \xi_2)$, and the range $x^1(\mathbb{R})$ is maximal in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly around ξ_1 with ranges in $(0, \xi_2)$. Therefore $\{\pi r_t : t \in \mathbb{R}\}$ either equals O_1 or belongs to $\operatorname{int}(O_1)$. Proposition 8.3 implies $V(x^{\varphi}_{t+u} - x^1_t) = 2$ for all $u, t \in \mathbb{R}$. With Lemma 2.3 (ii), this yields that the curve $S : \mathbb{R} \ni t \mapsto \pi x^{\varphi}_t \in \mathbb{R}^2$ does not intersect O_1. So necessarily r equals x^1 apart from shift and $\omega(\varphi) = \mathcal{O}_1$. In case there is $t_* \in \mathbb{R}$ such that $x^{\varphi}_{t_*} < 0$, we deduce in a similar way that $\omega(\varphi) = \mathcal{O}_{-1}$. □

Claim 8.9. Assume that for $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, the limit set $\omega(\varphi)$ is not a non-constant periodic orbit. Then it is a stable equilibrium.

Proof. As for all $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, the orbit $\{x^{\varphi}_t : t \geq 0\}$ is bounded, the Poincaré–Bendixson theorem can be applied (see Section 2). Hence if $\omega(\varphi)$ is not a non-constant periodic orbit, then for each $\psi \in \omega(\varphi)$, we have

$$\alpha(\psi) \cup \omega(\psi) \subset \left\{\hat{\xi}_i : \ i = -2, -1, 0, 1, 2\right\}.$$

If $\hat{\xi}_0$ is in $\omega(\varphi)$, then $\omega(\varphi) = \left\{\hat{\xi}_0\right\}$ as the equilibrium is stable. Similarly for $\hat{\xi}_{-2}$ and $\hat{\xi}_2$.

Suppose for contradiction that $\omega(\varphi)$ contains no stable equilibrium point. If φ is in the stable set of $\hat{\xi}_i$ with $i \in \{-1, 1\}$, then as (H2) holds, $V\left(x^{\varphi}_t - \hat{\xi}_i\right) > 2$ for all $t \in \mathbb{R}$ (see Section 2), a contradiction to Proposition 8.3. So there exists $\psi \in \omega(\varphi)$ such that ψ is not an equilibrium. Then $\alpha(\psi) \cup \omega(\psi) \subseteq \left\{\hat{\xi}_{-1}, \hat{\xi}_1\right\}$. As it is mentioned in Section 2, there exists no homoclinic orbit to $\hat{\xi}_1$ and to $\hat{\xi}_{-1}$. Hence $\alpha(\psi) \neq \omega(\psi)$. If $\alpha(\psi) = \left\{\hat{\xi}_{-1}\right\}$, then there is $t^* \in \mathbb{R}$ with $x^{\psi}_{t^*} \ll \hat{\xi}_0$. By Proposition 8.1, $x^{\psi}_t \ll \hat{\xi}_0$

for each $t > t^*$, a contradiction to $\omega(\psi) = \left\{\hat{\xi}_1\right\}$. One comes to the same conclusion assuming that $\alpha(\psi) = \left\{\hat{\xi}_1\right\}$ and $\omega(\psi) = \left\{\hat{\xi}_{-1}\right\}$. $\square$

It remains to show that the above connections indeed exist.

Recall that the unstable space

$$H_u = \{c_1 v_1 + c_2 v_2 : \; c_1, c_2 \in \mathbb{R}\}$$

of $DP(p_0)$ is 2-dimensional, where v_1 is a positive eigenfunction corresponding to the leading eigenvalue λ_1 and v_2 is the eigenfunction corresponding to the second eigenvalue λ_2 greater than one. Then for the solution $x_t^{v_2} : \mathbb{R} \to \mathbb{R}$ of the linear variational equation

$$\dot{x}(t) = -x(t) + f'(p(t-1))\, x(t-1) \tag{8.2}$$

with initial segment v_2, we have $V\left(x_t^{v_2}\right) = 2$ for all real t [17]. Clearly $v_2(-1) = 0$ and so $v_2(0) \neq 0$ by Lemma 2.3. Either $v_2(0) > 0$ or $v_2(0) < 0$ is possible. Assume eigenfunction v_2 is chosen so that $v_2(0) > 0$. Also, we may set $\|v_1\| = \|v_2\| = 1$.

For $n \geq 0$, let

$$S_n = \left\{\varphi \in C : \|\varphi - p_0\| = \frac{1}{n}\right\}$$

denote the sphere in C centered at p_0 with radius $1/n$. As $\mathcal{W}^u(p_0)$ and $\mathcal{W}_1^u(p_0)$ are 2-dimensional and 1-dimensional local manifolds tangent to $\{p_0\} + H_u$ and $\{p_0\} + H_u^1$ at p_0, respectively, there is $n_0 \geq 0$ such that for $n \geq n_0$, $S_n \cap \mathcal{W}^u(p_0)$ is homeomorphic to S^1, and in addition S_n and $\mathcal{W}_1^u(p_0)$ intersect in $\eta_1^n \in H$ and in $\eta_2^n \in H$. Based on the proof of Claim 8.7, we may suppose that $\eta_1^n \ll p_0 \ll \eta_2^n$ for each $n \geq n_0$, therefore $x_t^{\eta_1^n} \to \hat{\xi}_{-2}$ and $x_t^{\eta_2^n} \to \hat{\xi}_2$ as $t \to \infty$ for each $n \geq n_0$.

For $n \geq n_0$, let $C_n : [-1,1] \to S_n \cap \mathcal{W}^u(p_0)$ be a simple closed curve with $C_n(-1) = C_n(1) = \eta_1^n$ and $C_n(0) = \eta_2^n$ oriented so that $Pr_{H_u}(C_n(-1,0) - p_0)$ intersects $\{cv_2 : c < 0\} \subset H_u^2$ and $Pr_{H_u}(C_n(0,1) - p_0)$ intersects $\{cv_2 : c > 0\} \subset H_u^2$, see Fig. 13. This choice is possible. Obviously, $C_n(s) \neq p_0$ for all $n \geq n_0$ and $s \in [-1,1]$.

In order to prove the existence of the heteroclinic connections, we are going to apply the next assertion.

Claim 8.10. To each $\xi \in \{\xi_{-1}, \xi_0, \xi_1\}$, there correspond initial functions $\varphi \in \mathcal{W}^u(p_0)$ and $\psi \in \mathcal{W}^u(p_0)$ with

$$q(0) < \varphi(0) < p(0) < \psi(0) < 0$$

such that solutions $x^\varphi : \mathbb{R} \to \mathbb{R}$ and $x^\psi : \mathbb{R} \to \mathbb{R}$ oscillate around ξ.

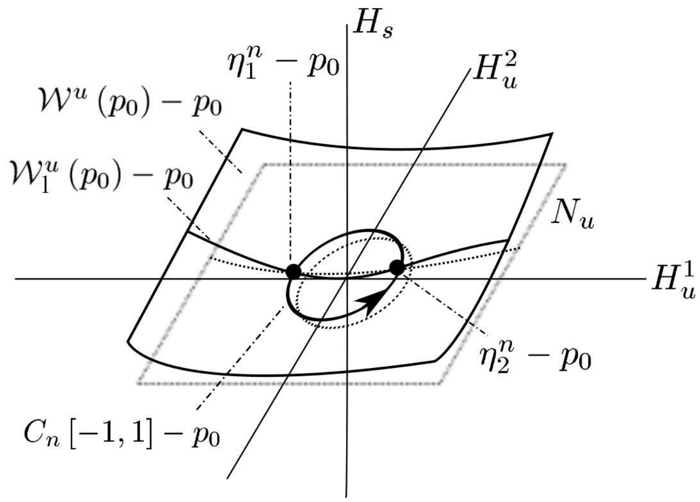

FIGURE 13. The unstable manifold and the image of C_n

Proof. Assume that $\xi \in \{\xi_{-1}, \xi_0, \xi_1\}$ and define

$$A_+ = \left\{\eta \in \mathcal{W}^u(p_0) : x_t^\eta \gg \hat{\xi} \text{ for some } t \geq 0\right\}$$

and

$$A_- = \left\{\eta \in \mathcal{W}^u(p_0) : x_t^\eta \ll \hat{\xi} \text{ for some } t \geq 0\right\}.$$

Clearly $\eta_1^n \in A_-$ and $\eta_2^n \in A_+$ for all $n \geq n_0$ because $x_t^{\eta_1^n} \to \hat{\xi}_{-2}$ and $x_t^{\eta_2^n} \to \hat{\xi}_2$ as $t \to \infty$. Then sets $A_+ \cap C_n[-1,0]$ and $A_- \cap C_n[-1,0]$ are disjoint, open and nonempty in $C_n[-1,0]$ for all $n \geq n_0$. It follows from connectedness that there exists $s_n \in (-1,0)$ with $C_n(s_n) \notin (A_+ \cup A_-)$, that is $x^{C_n(s_n)} : \mathbb{R} \to \mathbb{R}$ oscillates around ξ.

For $n \geq n_0$, the function $y^n : \mathbb{R} \to \mathbb{R}$ with

$$y^n(t) := \frac{x^{C_n(s_n)}(t) - p(t)}{\|C_n(s_n) - p_0\|}, \; t \in \mathbb{R},$$

satisfies the equation $\dot{y}^n(t) = -y^n(t) + a^n(t)\, y^n(t-1)$, where

$$a^n : \mathbb{R} \ni t \mapsto \int_0^1 f'\left(\theta x^{C_n(s_n)}(t-1) + (1-\theta)\, p(t-1)\right) \mathrm{d}\theta \in \mathbb{R}.$$

Because of the choice of curves C_n, $a^n(t) \to f'(p(t-1))$ as $n \to \infty$ uniformly on compact subsets of $[0,\infty)$.

Since $C_n(s_n) \in \mathcal{W}^u(p_0) \setminus \{p_0\}$ for all $n \geq n_0$, $C_n(s_n) \to p_0$ as $n \to \infty$ and $\mathcal{W}^u(p_0)$ is tangent to $\{p_0\} + H_u$ at p_0, we may suppose $y_0^n \to z_0 \in C$ as $n \to -\infty$, where $z_0 \in H_u$. Since $\|y_0^n\| = 1$ for all $n \geq n_0$, $\|z_0\| = 1$. Let $z : [-1,\infty) \to \mathbb{R}$ be the solution of (8.2) with initial data z_0. Then $y^n \to z$ uniformly on compact subsets of $[-1,\infty)$.

We claim that $z_0 = -v_2$. Assume that $z_0 = c_1 v_1 + c_2 v_2$ with $c_1 \neq 0$. As v_1 is a positive eigenfunction corresponding to the leading eigenvalue $\lambda_1 > 1$, there exists $t^* = t^*(c_1)$ such that $z_{t^*} \gg 0$ (or $z_{t^*} \ll 0$) and thus $y^n_{t^*} \gg 0$ (or $y^n_{t^*} \ll 0$) for some $n \geq n_0$. This is impossible by Proposition 8.3. So $z_0 = c_2 v_2$ with $c_2 \in \mathbb{R}$. The definition of C_n and the fact that $s_n \in (-1, 0)$ implies $c_2 \leq 0$. Also, $|c_2| = 1$ as $\|z_0\| = \|v_2\| = 1$. So $c_2 = -1$.

As $v_2(0) > 0$, we conclude that $z_0(0) < 0$. Since $y^n_0 \to z_0$ and $C_n(s_n) \to p_0$ as $n \to \infty$, there exist $n_1 \in \mathbb{N}$ so that for $n \geq n_1$, $q_0(0) < C_n(s_n)(0) < p_0(0)$. Accordingly set $\varphi = C_{n_1}(s_{n_1})$.

Similarly, there exists $t_n \in (0, 1)$ so that solution $x^{C_n(t_n)} : \mathbb{R} \to \mathbb{R}$ oscillates around ξ. The same reasoning carried out for $(C_n(t_n))^{\infty}_{n_0}$ instead of $(C_n(s_n))^{\infty}_{n_0}$ implies that $p_0(0) < C_n(t_n)(0) < 0$ for all $n \geq n_2$ with some $n_2 \in \mathbb{N}$. So choose $\psi = C_{n_2}(t_{n_2})$.

Clearly φ and ψ are in possession of the required properties. $\square$

Claim 8.11. There exist heteroclinic connections from $\mathcal{O}_p$ to $\hat{0}$ and to $\mathcal{O}_q$.

Proof. Claim 8.10 gives that there exists $\eta_3, \eta_4 \in \mathcal{W}^u(p_0)$ with

$$q(0) < \eta_3(0) < p(0) < \eta_4(0) < 0$$

such that solutions $x^{\eta_3} : \mathbb{R} \to \mathbb{R}$ and $x^{\eta_4} : \mathbb{R} \to \mathbb{R}$ oscillate around 0. Claim 8.9 gives that $\omega(\eta_i)$, $i \in \{3, 4\}$, is either a periodic orbit or a stable equilibrium. If $\omega(\eta_3) = \left\{\hat{\xi}_2\right\}$, then by the monotone property of the semiflow Φ (see Proposition 8.1) there is $t_0 > 0$ such that $x^{\eta_3}_t \gg 0$ for $t > t_0$, a contradiction. Similarly, $\omega(\eta_3) \neq \left\{\hat{\xi}_{-2}\right\}$ and $\omega(\eta_4) \nsubseteq \left\{\hat{\xi}_{-2}, \hat{\xi}_2\right\}$. We prove that $\omega(\eta_3) = \mathcal{O}_q$ and $\omega(\eta_4) = \{\hat{0}\}$.

Consider the curves

$$S_3 : \mathbb{R} \ni t \mapsto \pi x^{\eta_3}_t \in \mathbb{R}^2 \text{ and } S_4 : \mathbb{R} \ni t \mapsto \pi x^{\eta_4}_t \in \mathbb{R}^2.$$

By Corollary 8.4, S_3 and S_4 are simple, furthermore they have no points in common with O_p.

Function η_3 is selected so that $S_3(0) = (\eta_3(0), \eta_3(-1)) \in \operatorname{ext}(O_p)$. Thus $S_3(t) \in \operatorname{ext}(O_p)$ for all $t \in \mathbb{R}$. As a consequence, $\hat{0}$ is not in $\omega(\eta_3)$. Note that all the other stable equilibria have already been excluded, hence it follows from Claim 8.9 that $\omega(\eta_3) = \{r_t : t \in \mathbb{R}\}$, where r is a nontrivial periodic solution of Eq. (1.1). As x^{η_3} oscillates around 0, $\omega(\eta_3) = \mathcal{O}_q$ by Claim 8.8.

Similarly, Claim 8.8 yields that if $\omega(\eta_4)$ is a non-constant periodic orbit, then $\omega(\eta_4) = \mathcal{O}_q$. However, the choice of η_4 implies $S_4(0) = (\eta_3(0), \eta_3(-1))^{tr} \in \operatorname{int}(O_p)$, hence $S_4(t) \in \operatorname{int}(O_p)$ for all $t \in \mathbb{R}$. It follows immediately that $\omega(\eta_4) \neq \mathcal{O}_q$. So $\omega(\eta_4)$ is a stable equilibrium by Claim 8.9. As $\hat{\xi}_{-2}$ and $\hat{\xi}_2$ have been excluded at the beginning of the proof, necessarily $\omega(\eta_4) = \{\hat{0}\}$. □

Claim 8.12. There are heteroclinic connections from $\mathcal{O}_p$ to the orbits $\mathcal{O}_1$ and $\mathcal{O}_{-1}$.

Proof. According to Claim 8.10, there exists $\eta_5 \in \mathcal{W}^u(p_0)$ with $0 > \eta_5(0) > p(0)$ such that solution $x^{\eta_5} : \mathbb{R} \to \mathbb{R}$ oscillates around ξ_1. Curve $S_5 : \mathbb{R} \ni t \mapsto \pi x_t^{\eta_5} \in \mathbb{R}^2$ does not intersect O_p. Hence $S(t) \in \operatorname{int}(O_p)$ for all $t \in \mathbb{R}$ and $\omega(\eta_5) \neq \mathcal{O}_q$. Also, $\omega(\eta_5)$ is not a stable equilibrium or $\mathcal{O}_{-1}$ as x^{η_5} oscillates around ξ_1. So $\omega(\eta_5) = \mathcal{O}_1$, see Claim 8.8.

At last, set $\eta_6 \in \mathcal{W}^u(p_0)$ with $0 > \eta_6(0) > p(0)$ so that $x^{\eta_6} : \mathbb{R} \to \mathbb{R}$ oscillates around ξ_{-1}. This is possible by Claim 8.10. An analogous argument verifies that $\omega(\eta_6) = \mathcal{O}_{-1}$. □

References

[1] Computer Assisted Proofs in Dynamics group, a C++ package for rigorous numerics, http://capd.wsb-nlu.edu.pl.

[2] Chen Y., Krisztin T., and Wu J. (2000). Connecting orbits from synchronous periodic solutions in phase-locked periodic solutions in a delay differential system. J. Differential Equations 163, 130–173.

[3] Chen, Y., and Wu, J. (2001). Existence and attraction of a phase-locked oscillation in a delayed network of two neurons. Differential Integral Equations 14, 1181–1236.

[4] Chen, Y., and Wu, J. (1999). Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network. Phys. D 134, 185–199.

[5] Chen Y., Yi T., and Wu J. (2010). Periodic solutions and the global attractor in a system of delay differential equations. SIAM J. Math. Anal. 42, 24–63.

[6] Diekmann, O., van Gils, S. A., Verduyn Lunel, S. M., and Walther, H.-O. (1995). Delay equations. Functional, complex, and nonlinear analysis, Springer-Verlag, New York.

[7] Erneux, T. (2009). Applied delay differential equations, Springer, New York.

[8] Gerstner,W., and Kistler, W. M. (2002). Spiking neuronmodels, single neurons, populations, plasticity, Cambridge University Press, Cambridge.

[9] Hale, J. K. (1988). Asymptotic behavior of dissipative systems, American Mathematical Society, Providence, RI.

[10] Hale J. K. (1969). Ordinary differential equations, Interscience, New York.

[11] Hale, J. K., and Verduyn Lunel, S. M. (1993). Introduction to functional-differential equations, Springer-Verlag, New York.

[12] Kennedy, B., Stability and instability for periodic solutions of delay equations with steplike feedback. In preparation.

[13] Krisztin, T. (2008). Global dynamics of delay differential equations. Period. Math. Hungar. 56, 83–95.

[14] Krisztin, T. (2001). Unstable sets of periodic orbits and the global attractor for delayed feedback. In Faria T., Freitas P. (ed.), Topics in functional differential and difference equations, Amer. Math. Soc., Providence, RI, pp. 267–296.

[15] Krisztin, T. (2000). The unstable set of zero and the global attractor for delayed monotone positive feedback. Discrete Contin. Dynam. Systems, Added Volume, 229–240.

[16] Krisztin T., and Walther H.-O. (2001). Unique periodic orbits for delayed positive feedback and the global attractor. J. Dynam. Differential Equations 13, 1–57.

[17] Krisztin T., Walter H.-O., and Wu J. (1999). Shape, smoothness and invariant stratification of an attracting set for delayed monotone positive feedback, Amer. Math. Soc., Providence, RI.

[18] Krisztin T., and Wu J. The global structure of an attracting set. In preparation.

[19] Lani-Wayda B. (1995). Persistence of Poincaré mappings in functional differential equations (with application to structural stability of complicated behavior). J. Dynam. Differential Equations **7**, 1–71.

[20] Mallet-Paret, J. (1988). Morse decompositions for delay-differential equations. J. Differential Equations 72, 270–315.

[21] Mallet-Paret, J., and Sell G. R. (1996). Systems of differential delay equations: Floquet multipliers and discrete Lyapunov Functions. J. Differential Equations 125, 385–440.

[22] Mallet-Paret, J., and Sell, G. R. (1996). The Poincaré–Bendixson theorem for monotone cyclic feedback systems with delay. J. Differential Equations 125, 441–489.

[23] McCord, C., and Mischaikow, K. (1996). On the global dynamics of attractors for scalar delay equations. J. Amer. Math. Soc. 9, 1095–1133.

[24] Polner, M. (2002). Morse decomposition for delay differential equations with positive feedback. Nonlinear Anal. 48, 377-397.

[25] Sard, A. (1942). The measure of the critical values of differentiable maps. Bull. Amer. Math. Soc. 48, 883–890.

[26] Smith, H. L. (1995). Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Amer. Math. Soc., Providence, RI.

[27] Stoffer, D. (2008). Delay equations with rapidly oscillating stable periodic solutions. J. Dynam. Differential Equations 20, 201–238.

[28] Walther, H.-O., (2001). Contracting return maps for some delay differential equations. In Faria T., Freitas P. (ed.), Topics in functional differential and difference equations, Amer. Math. Soc., Providence, RI, pp. 349–360.

[29] Walther, H.-O. (1995). The 2-dimensional attractor of $x'(t) = -\mu x(t) + f(x(t-1))$. Mem. Amer. Math. Soc. 113, no. 544.

[30] Walther, H.-O. (1998). The singularities of an attractor of a delay differential equation. Funct. Differ. Equ. 5, 513–548.

[31] Wu, J. (2001). Introduction to neural dynamics and signal transmission delay, Walter de Gruyter & Co., Berlin.