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C^1-Smoothness of Center Manifolds for Differential Equations with State-dependent Delay

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Abstract. In the survey paper [4] a Lipschitz continuous local center manifold is constructed at a stationary point for a class of differential equations with state-dependent delay. Here we show that the obtained center manifold is continuously differentiable.

1 Introduction

Recently, for a class of functional differential equations including equations with state-dependent delays, Walther [9, 10] developed the fundamental theory. This class of equations has in general less smoothness properties than those representing equations with constant delay, and the classical theory (see, e.g., [2, 3]) is not applicable. Walther [9, 10] introduced the so called solution manifold, a smooth submanifold of finite codimension of a function space, and proved under mild smoothness hypotheses that the initial value problem is well-posed on the solution manifold, and the solutions define a semiflow of continuously differentiable solution operators. In addition, Walther resolved the problem of linearization for equations with state-dependent delay by demonstrating that the earlier heuristic linearization technique (see, e.g., [1]) is the true linearization in his framework.

Under the same hypotheses, at stationary points the continuously differentiable solution operators have local stable, center and unstable manifolds. It is shown in the survey paper [4] that these stable and unstable manifolds of maps yield local stable and unstable manifolds also for the semiflow. The same approach does not immediately work for center manifolds since the local center manifolds of the solution operators (or time-t maps) are not necessarily invariant under the semiflow, see [6].

The survey paper [4] constructed Lipschitz continuous local center manifolds for the semiflow at stationary points generated by a class of functional differential equations representing equations with state-dependent delays. The aim of this work

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is to prove that these center manifolds are continuously differentiable. In Section 2 we recall certain steps from the construction of [4] and the necessary technical tools. The proof of the smoothness is contained in Section 3. The proof applies the Lyapunov–Perron approach and closely follows that of [2]. However, as the right hand side of the equation has smoothness properties only on the space of continuously differentiable functions instead of the space of continuous functions as in the classical case ([2, 3]), the space, where the fixed point problem is formulated, is different, and some technical parts are also different. The smoothness proof is based on a slight modification (proved in [7]) of a result of Vanderbauwhede and van Gils [8] on contractions on embedded Banach spaces.

We remark that only C^1-smoothness of the semiflow is shown by Walther [9, 10], and as far as we know, more smoothness is a problem. It is also an open problem to construct center manifolds with more smoothness properties. This would be important in local bifurcation theory through the center manifold reduction. It is interesting that the paper [5] contains a result about C^k-smooth local unstable manifolds with $k \geq 1$ without using any smoothness property of the semiflow.

For motivation, applications and several additional results for functional differential equations with state-dependent delays we refer to the survey paper [4].

2 Preliminaries

Let $|\cdot|$ be a norm in $\mathbb{R}^n$. Fix an $h > 0$. The Banach spaces of continuous and continuously differentiable maps $\phi : [-h, 0] \to \mathbb{R}^n$ are denoted by

$$C = C([-h, 0], \mathbb{R}^n) \quad \text{and} \quad C^1([-h, 0], \mathbb{R}^n),$$

respectively. The norms are given by

$$|\phi|_C = \max_{-h \leq s \leq 0} |\phi(s)| \quad \text{and} \quad |\phi|_{C^1} = |\phi|_C + |\phi'|_C,$$

respectively. If I is an interval, $x : I \to \mathbb{R}^n$ is a function, and $t \in I$ with $t - h \in I$, then the segment $x_t : [-h, 0] \to \mathbb{R}^n$ is defined by

$$x_t(s) = x(t + s), \qquad -h \leq s \leq 0.$$

Let an open subset $U \subset C^1$ and a map $f : U \to \mathbb{R}^n$ be given. Throughout this paper, it is assumed that

(S1): f is continuously differentiable,

(S2): each derivative $Df(\phi)$, $\phi \in U$, extends to a linear continuous map $D_e f(\phi) : C \to \mathbb{R}^n$, and

(S3): the map

$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Consider the functional differential equation

$$x'(t) = f(x_t) \tag{2.1}$$

with the initial condition

$$x_0 = \phi \in U. \tag{2.2}$$

We refer to [9] to see that, for example, if $g \in C^1(\mathbb{R}, \mathbb{R}^n)$, and for $\tau : U \to [0, h]$ hypotheses (S1)-(S3) hold with $n = 1$, then

$$f(\phi) = g(\phi(-\tau(\phi))), \qquad \phi \in U,$$

satisfies (S1)-(S3). Thus, Equation (2.1) contains state-dependent delay differential equations of the form

$$x'(t) = g(x(t - \tau(x_t))).$$

By a solution of (2.1)-(2.2) we understand a continuously differentiable function $x : [-h, t_*) \to \mathbb{R}^n$, $0 < t_* \leq \infty$, satisfying $x_t \in U$, $0 \leq t < t_*$, $x_0 = \phi$, and (2.1) for $0 < t < t_*$.

The closed subset

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

of U is called the solution manifold of (2.1). In the sequel we assume $X_f \neq \emptyset$. The papers [9, 10] contain the following basic results. X_f is a C^1-smooth submanifold of U with codimension n. Each $\phi \in X_f$ uniquely defines a noncontinuable solution $x^\phi : [-h, t_+(\phi)) \to \mathbb{R}^n$ of (2.1)-(2.2). All segments x_t^ϕ, $0 \leq t < t_+(\phi)$, belong to X_f, and the relations

$$F(t, \phi) = x_t^\phi, \ \phi \in X_f, \ 0 \leq t < t_+(\phi),$$

define a continuous semiflow $F : \Omega \to X_f$, where $\Omega = \{(t, \phi) : \phi \in X_f, \ 0 \leq t < t_+(\phi)\}$. Each map

$$F(t, \cdot) : \{\phi \in X_f : (t, \phi) \in \Omega\} \to X_f$$

is continuously differentiable, and for all $(t, \phi) \in \Omega$ and $\chi \in T_\phi X_f$ we have

$$D_2 F(t, \phi)\chi = v_t^{\phi, \chi}$$

with the solution $v^{\phi,\chi} : [-h, t_+(\phi)) \to \mathbb{R}^n$ of the linear initial value problem

$$v'(t) = Df(F(t, \phi))v_t, \qquad v_0 = \chi.$$

The tangent spaces of the manifold X_f are

$$T_\phi X_f = \left\{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\right\}.$$

Assume that $0 \in U$ and 0 is a stationary point of F, i.e., $f(0) = 0$. The linearization of F at 0 is the strongly continuous semigroup $T(t) = D_2 F(t, 0)$, $t \geq 0$, on the Banach space

$$T_0 X_f = \left\{\chi \in C^1 : \chi'(0) = Df(0)\chi\right\}$$

equipped with the norm $|\cdot|_{C^1}$.

The solutions of the linear initial value problem

$$y'(t) = D_e f(0) y_t, \ y_0 = \phi \in C$$

define the strongly continuous semigroup $T_e(t)$, $t \geq 0$, on C. The generator of T_e is $G_e : \mathrm{dom}(G_e) \to C$ with $\mathrm{dom}(G_e) = T_0 X_f$ and $G_e \phi = \phi'$.

Let G denote the generator of $T(t)$, $t \geq 0$. By [4] the domain of G is

$$\left\{\chi \in C^2 : \chi \in \mathrm{dom}(G_e), \ \chi' \in \mathrm{dom}(G_e)\right\},$$

and $G\chi = \chi'$. For the spectra of G_e and G,

$$\sigma(G_e) = \sigma(G)$$

holds. All points of $\sigma(G_e)$ are eigenvalues with finite dimensional generalized eigenspaces. The eigenvalues coincide with the zeros of the characteristic function

$$\mathbb{C} \ni z \mapsto \det(zI - D_e f(0) e^{z\cdot}) \in \mathbb{C}.$$

The realified generalized eigenspaces of G_e given by the eigenvalues with negative, zero and positive real part are the stable C_s, center C_c and unstable C_u spaces, respectively. We have the decomposition

$$C = C_s \oplus C_c \oplus C_u,$$

C_s is infinite dimensional, C_c and C_u are finite dimensional, $C_c \subset \mathrm{dom}(G_e)$, $C_u \subset \mathrm{dom}(G_e)$. The set $C_s^1 = C_s \cap C^1$ is a closed subset of C^1. Hence the decomposition

$$C^1 = C_s^1 \oplus C_c \oplus C_u \tag{2.3}$$

of C^1 holds as well.

The stable, center and unstable subspaces of G are $C_s \cap \mathrm{dom}(G_e) = C_s \cap \mathrm{dom}(G_e)$, C_c and C_u, respectively, and

$$T_0 X_f = \mathrm{dom}(G_e) = (C_s \cap \mathrm{dom}(G_e)) \oplus C_c \oplus C_u.$$

In the sequel we assume

$$\sigma(G_e) \cap i\mathbb{R} \neq \emptyset,$$

that is, $\dim C_c \geq 1$.

In [4] we constructed a Lipschitz continuous (local) center manifold of F at the stationary point 0.

Theorem A *There exist open neighbourhoods $C_{c,0}$ of 0 in C_c and $C^1_{su,0}$ of 0 in $C_s^1 \oplus C_u$ with $N = C_{c,0} + C^1_{su,0} \subset U$, a Lipschitz continuous map $w_c : C_{c,0} \to C^1_{su,0}$ such that $w_c(0) = 0$ and for the graph*

$$W_c = \{\phi + w_c(\phi) : \phi \in C_{c,0}\}$$

of w_c the following hold.

(i) *$W_c \subset X_f$, and W_c is a $\dim C_c$-dimensional Lipschitz smooth submanifold of X_f.*
(ii) *If $x : \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable solution of (2.1) on $\mathbb{R}$ with $x_t \in N$ for all $t \in \mathbb{R}$, then $x_t \in W_c$ for all $t \in \mathbb{R}$.*
(iii) *W_c is locally positively invariant with respect to the semiflow F, i.e., if $\phi \in W_c$ and $\alpha > 0$ such that $F(t, \phi)$ is defined for all $t \in [0, \alpha)$, and $F(t, \phi) \in N$ for all $t \in [0, \alpha)$, then $F(t, \phi) \in W_c$ for all $t \in [0, \alpha)$.*

The aim of this paper is to show the following

Theorem 2.1 *The map $w_c : C_{c,0} \to C^1_{su,0}$ is continuously differentiable, and $Dw_c(0) = 0$.*

The proof of Theorem A in [4] applies the Lyapunov–Perron approach which is based on a variation-of-constants formula of [2]. We recall some basic facts from [2], and also some steps from the proof of Theorem A.

We denote dual spaces and adjoint operators by an asterisk $*$ in the sequel. The elements $\phi^\odot$ of C^* for which the curve

$$[0, \infty) \ni t \mapsto T_e^*(t)\phi^\odot \in C^*$$

is continuous form a closed subspace $C^\odot$ (of C^*) which is positively invariant under $T_e^*(t)$, $t \geq 0$. The operators

$$T_e^\odot(t) : C^\odot \ni \phi^\odot \mapsto T_e^*(t)\phi^\odot \in C^\odot, \ t \geq 0,$$

constitute a strongly continuous semigroup on $C^\odot$. Similarly, we can introduce the dual space $C^{\odot *}$ and the semigroup of adjoint operators $T_e^{\odot *}(t)$, $t \geq 0$, which is strongly continuous on $C^{\odot\odot}$. There is an isometric isomorphism between $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ equipped with the norm $|(\alpha,\phi)| = \max\{|\alpha|, |\phi|_\infty\}$ and $C^{\odot *}$, where $L^\infty(-h,0;\mathbb{R}^n)$ denotes the Banach space of measurable and essentially bounded functions from $[-h,0]$ into $\mathbb{R}^n$ equipped with the L^∞-norm $|\cdot|_\infty$. We will identify $C^{\odot *}$ with $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ and omit the isomorphism. The original state space is sun-reflexive in the sense that, for the norm-preserving linear map $\iota : C \to C^{\odot *}$ given by $\iota(\phi) = (\phi(0),\phi)$, we have $\iota(C) = C^{\odot\odot}$. We also omit the embedding operator ι and identify C and $C^{\odot\odot}$. All of these results as well as the decomposition of $C^{\odot *}$ and the variation-of-constants formula can be found in [2].

Let $Y^{\odot *}$ denote the subspace $\mathbb{R}^n \times \{0\}$ of $C^{\odot *}$. For the k-th unit vector e_k in $\mathbb{R}^n$ set $r_k^{\odot *} = (e_k, 0) \in Y^{\odot *}$. Let $l : \mathbb{R}^n \to Y^{\odot *}$ be the linear map given by $l(e_k) = r_k^{\odot *}$, $k \in \{1,2,\ldots,n\}$. Then l has an inverse l^{-1}, and $|l| = |l^{-1}| = 1$.

Let $G_e^{\odot *}$ denote the generator of $T_e^{\odot *}$. For the spectra $\sigma(G_e)$ and $\sigma(G_e^{\odot *})$ we have $\sigma(G_e) = \sigma(G_e^{\odot *})$. Recall that we assumed $\sigma(G_e) \cap i\mathbb{R} \neq \emptyset$. Then $C^{\odot *}$ can be decomposed as

$$C^{\odot *} = C_s^{\odot *} \oplus C_c \oplus C_u, \tag{2.4}$$

where $C_s^{\odot *}$, C_c, C_u are closed subspaces of $C^{\odot *}$, C_c and C_u are contained in C^1, $1 \leq \dim C_c < \infty$, $\dim C_u < \infty$. The subspaces $C_s^{\odot *}$, C_c and C_u are invariant under $T_e^{\odot *}(t)$, $t \geq 0$, and $T_e(t)$ can be extended to a one-parameter group on both C_c and C_u. There exist real numbers $K \geq 1$, $a < 0$, $b > 0$ and $\epsilon > 0$ with $\epsilon < \min\{-a, b\}$ such that

$$\begin{aligned} |T_e(t)\phi| &\leq K e^{bt}|\phi|, && t \leq 0,\ \phi \in C_u, \\ |T_e(t)\phi| &\leq K e^{\epsilon|t|}|\phi|, && t \in \mathbb{R},\ \phi \in C_c, \\ |T_e^{\odot *}(t)\phi| &\leq K e^{at}|\phi|, && t \geq 0,\ \phi \in C_s^{\odot *}. \end{aligned} \tag{2.5}$$

Using the identification of C and $C^{\odot\odot}$, we obtain $C_s^1 = C^1 \cap C_s^{\odot *}$. The decompositions (2.3) and (2.4) define the projection operators P_s, P_c, P_u and $P_s^{\odot *}, P_c^{\odot *}, P_u^{\odot *}$ with ranges C_s^1, C_c, C_u and $C_s^{\odot *}, C_c, C_u$, respectively. P_{su}^1 denotes the projection of C^1 onto $C_s^1 \oplus C_u$ along C_c.

We need a variation-of-constants formula for solutions of

$$x'(t) = D_e f(0) x_t + q(t) \tag{2.6}$$

with a continuous function $q : \mathbb{R} \to \mathbb{R}^n$.

If c, d are reals with $c \leq d$, and $w : [c,d] \to C^{\odot *}$ is continuous, then the weak-star integral

$$\int_c^d T_e^{\odot *}(d-\tau) w(\tau)\, d\tau \in C^{\odot *}$$

is defined by

$$\left(\int_c^d T_e^{\odot *}(d-\tau) w(\tau)\, d\tau \right)(\phi^\odot) = \int_c^d T_e^{\odot *}(d-\tau) w(\tau)(\phi^\odot)\, d\tau$$

for all $\phi^\odot \in C^\odot$.

If $I \subset \mathbb{R}$ is an interval, $q : I \to \mathbb{R}^n$ is continuous and $x : I + [-h,0] \to \mathbb{R}^n$ is a solution of (2.6) on I, that is, x is continuous on $I + [-h,0]$, continuously differentiable on I, and (2.6) holds for all $t \in I$, then the curve $u : I \ni t \mapsto x_t \in C$ satisfies the integral equation

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau) Q(\tau)\, d\tau, \qquad t, s \in I,\ s \leq t, \tag{2.7}$$

with $Q(t) = l(q(t))$, $t \in I$. Moreover, if $Q : I \to Y^{\odot *}$ is continuous, and a continuous $u : I \to C$ satisfies (2.7), then there is a continuous function $x : I + [-h, 0] \to \mathbb{R}^n$ such that $x_t = u(t)$ for all $t \in I$, x is continuously differentiable on I, and x satisfies (2.6) with $q(t) = l^{-1}(Q(t))$, $t \in I$. So, there is a one-to-one correspondence between the solutions of (2.6) and (2.7).

For a Banach space B with norm $|\cdot|$ and a real $\eta \geq 0$, define the Banach space

$$C_\eta(\mathbb{R}, B) = \left\{ b \in C(\mathbb{R}, B) : \sup_{t \in \mathbb{R}} e^{-\eta |t|} |b(t)| < \infty \right\}$$

with norm

$$|b|_{C_\eta(\mathbb{R}, B)} = \sup_{t \in \mathbb{R}} e^{-\eta |t|} |b(t)|.$$

For $\eta \geq 0$, we introduce the notation

$$Y_\eta = C_\eta(\mathbb{R}, Y^{\odot *}), \;\; C^0_\eta = C_\eta(\mathbb{R}, C), \;\; C^1_\eta = C_\eta(\mathbb{R}, C^1).$$

For a given $Q : \mathbb{R} \to Y^{\odot *}$ we (formally) define

$$(\mathcal{K}Q)(t) = \int_0^t T_e^{\odot *}(t - \tau) P_c^{\odot *} Q(\tau)\, d\tau + \int_\infty^t T_e^{\odot *}(t - \tau) P_u^{\odot *} Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t - \tau) P_s^{\odot *} Q(\tau)\, d\tau.$$

The weak-star integrals over unbounded intervals are defined as for finite intervals above, see the details in [2].

The results in the remaining part of this section are either shown in [4] or can be obtained in a straightforward manner.

For any $\eta \in (\epsilon, \min\{-a, b\})$,

$$\mathcal{K}(Y_\eta) \subset C^1_\eta,$$

and the induced linear map $\mathcal{K}_\eta : Y_\eta \to C^1_\eta$ is linear bounded with

$$|\mathcal{K}_\eta| \leq c(\eta),$$

where

$$c(\eta) = K\left(1 + e^{\eta h} |D_e f(0)|\right) \left(\frac{1}{\eta - \epsilon} + \frac{1}{-a - \eta} + \frac{1}{b - \eta} \right) + e^{\eta h}.$$

Moreover, if $Q \in Y_\eta$ then $u = \mathcal{K}Q$ is the unique solution of

$$u(t) = T_e(t - s) u(s) + \int_s^t T_e^{\odot *}(t - \tau) Q(\tau)\, d\tau, \qquad -\infty < s \leq t < \infty, \tag{2.8}$$

in C^1_η with $P_c^{\odot *} u(0) = 0$.

As $\dim C_c < \infty$, there is a norm $|\cdot|_c$ on C_c which is C^∞-smooth on $C_c \setminus \{0\}$. Then

$$|\phi|_1 = \max\{|P_c \phi|_c, |P^1_{su} \phi|_{C^1}\}, \qquad \phi \in C^1,$$

defines the new norm $|\cdot|_1$ on C^1 which is equivalent to $|\cdot|_{C^1}$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function so that $\rho(t) = 1$ for $t \leq 1$, $\rho(t) = 0$ for $t \geq 2$, and $\rho(t) \in (0, 1)$ for $t \in (1, 2)$.

Define $r : U \ni \phi \mapsto f(\phi) - D_e f(0) \phi \in \mathbb{R}^n$ and

$$\hat{r}(\phi) = \begin{cases} r(\phi), & \text{if } \phi \in U; \\ 0, & \text{if } \phi \notin U. \end{cases}$$

For any $\delta > 0$, let

$$r_\delta(\phi) = \hat{r}(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right)\rho\left(\frac{|P^1_{su}\phi|_1}{\delta}\right), \qquad \phi \in C^1.$$

Clearly, $r_\delta : C^1 \to \mathbb{R}^n$ is continuous. For $\gamma > 0$ set $B_\gamma(C^1) = \{\phi \in C^1 : |\phi|_1 < \gamma\}$.

Choose $\delta_0 > 0$ so that

$$B_{2\delta_0}(C^1) \subset U,$$

and $r|_{B_{2\delta_0}(C^1)}$, $Dr|_{B_{2\delta_0}(C^1)}$ are bounded. Then, for any $\delta \in (0, \delta_0)$,

$$r_\delta|_{\{\phi\in C^1:|P^1_{su}\phi|_1<\delta\}}(\phi) = \hat{r}(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right), \qquad \phi \in C^1,$$

and $r_\delta|_{\{\phi\in C^1:|P^1_{su}\phi|_1<\delta\}}$ is a bounded and C^1-smooth function with bounded derivative.

There exist $\delta_1 \in (0, \delta_0)$ and a nondecreasing function $\mu : [0, \delta_1] \to [0, 1]$ such that μ is continuous at 0, $\mu(0) = 0$, and for all $\delta \in (0, \delta_1]$ and for all $\phi, \psi \in C^1$

$$\begin{aligned} |r_\delta(\phi)| &\le \delta\mu(\delta), \\ |r_\delta(\phi) - r_\delta(\psi)| &\le \mu(\delta)|\phi - \psi|_{C^1}. \end{aligned} \tag{2.9}$$

For $\delta \in (0, \delta_1]$ we consider the modified equations

$$x'(t) = D_e f(0)x_t + r_\delta(x_t), \qquad t \in \mathbb{R}, \tag{2.10}$$

and

$$u(t) = T_e(t - s)u(s) + \int_s^t T_e^{\odot *}(t - \tau)l(r_\delta(u(\tau)))\,d\tau, \qquad -\infty < s \le t < \infty. \tag{2.11}$$

These equations are equivalent in the following sense: If $x : \mathbb{R} \to \mathbb{R}^n$ is C^1-smooth and is a solution of Equation (2.10), then $u : \mathbb{R} \ni t \mapsto x_t \in C^1$ is a solution of Equation (2.11), and conversely, a continuous $u : \mathbb{R} \to C^1$ satisfying (2.11) defines a C^1-smooth solution of (2.10) by $x(t) = u(t)(0)$, $t \in \mathbb{R}$.

As in [4] we fix an $\eta_0 \in (\epsilon, \min\{-a, b\})$ and a $\delta \in (0, \delta_1)$ such that

$$c(\eta_0)\mu(\delta) < \frac{1}{2}.$$

By the continuity of $c(\eta)$ for $\epsilon < \eta < \min\{-a, b\}$, there is $\eta_1 \in (\eta_0, \min\{-a, b\})$ such that

$$c(\eta)\mu(\delta) < \frac{1}{2} \qquad \text{for all } \eta \in [\eta_0, \eta_1]$$

also holds.

Define the substitution operator

$$R : (C^1)^{\mathbb{R}} \to (Y^{\odot *})^{\mathbb{R}}$$

by

$$R(u)(t) = l(r_\delta(u(t))).$$

Then, for each $\eta > 0$, we have

$$R(C^1_\eta) \subset Y_\eta.$$

By using inequalities (2.9) and $|l| = 1$, it follows that for the induced maps

$$R_\eta : C^1_\eta \to Y_\eta$$

the inequalities

$$|R_\eta(u)|_{Y_\eta} \le \delta\mu(\delta),$$
$$|R_\eta(u) - R_\eta(v)|_{Y_\eta} \le \mu(\delta)|u - v|_{C^1_\eta}$$

hold for all $u \in C^1_\eta$, $v \in C^1_\eta$ and $\eta \in [\eta_0, \eta_1]$.

Define the mapping $S : C_c \to C^1_\eta$ by

$$(S\phi)(t) = T_e(t)\phi.$$

If $\eta > \epsilon$ then $S(C_c) \subset C^1_\eta$, and the induced map $S_\eta : C_c \to C^1_\eta$ is bounded linear with $|S_\eta| \le K$.

For each $\eta \in [\eta_0, \eta_1]$ a mapping

$$G_\eta : C^1_\eta \times C_c \to C^1_\eta$$

can be defined by

$$G_\eta(u, \phi) = S_\eta(\phi) + \mathcal{K}_\eta \circ R_\eta(u).$$

For all $u, v \in C^1_\eta$, $\phi \in C_c$ and $\eta \in [\eta_0, \eta_1]$, this mapping satisfies

$$\begin{aligned}|G_\eta(u, \phi) - G_\eta(v, \phi)|_{C^1_\eta} &\le |\mathcal{K}_\eta|\,|R_\eta(u) - R_\eta(v)|_{Y_\eta} \\ &\le c(\eta)\mu(\delta)|u - v|_{C^1_\eta} \\ &\le \frac{1}{2}|u - v|_{C^1_\eta}.\end{aligned}$$

So for every $\phi \in C_c$ there is a unique fixed point $u_\eta(\phi) \in C^1_\eta$ of the contractions $G_\eta(\cdot, \phi) : C^1_\eta \to C^1_\eta$, $\eta \in [\eta_0, \eta_1]$. We have

$$P_c u_\eta(\phi)(0) = (S_\eta(\phi))(0) = \phi.$$

Therefore $u \in C^1_\eta$ is a solution of Equation (2.11) with $P_c u(0) = \phi \in C_c$ if and only if $u = u_\eta(\phi)$. The maps $u_\eta : C_c \to C^1_\eta$, $\eta_0 \le \eta \le \eta_1$, are Lipschitz continuous, and $u_\eta(0) = 0$.

Introduce the maps

$$w_\eta : C_c \ni \phi \mapsto P^1_{su} u_\eta(\phi)(0) \in C^1_s \oplus C_u$$

for $\eta_0 \le \eta \le \eta_1$. The sets

$$W_\eta = \{u_\eta(\phi)(0) : \phi \in C_c\} = \{\phi + w_\eta(\phi) : \phi \in C_c\}$$

are called the global center manifolds of the modified equation (2.11) at the stationary point 0, $\eta_0 \le \eta \le \eta_1$.

An important observation of [4] was that, for each $\phi \in C_c$ and $\eta \in [\eta_0, \eta_1]$,

$$\begin{aligned}|w_\eta(\phi)|_1 &= |P^1_{su} u_\eta(\phi)(0)|_{C^1} \\ &= |\mathcal{K}_\eta(R_\eta(u_\eta(\phi)))(0)|_{C^1} \le |\mathcal{K}_\eta(R_\eta(u_\eta(\phi)))|_{C^1_\eta} \\ &\le |\mathcal{K}_\eta|\,|R_\eta(u_\eta(\phi))|_{Y_\eta} \\ &\le c(\eta)\delta\mu(\delta) < \delta.\end{aligned}$$

It is also observed in [4] that for all $t \in \mathbb{R}$,

$$u_\eta(\phi)(t) = u_\eta(P_c u_\eta(\phi)(t))(0), \qquad \phi \in C_c,\ \eta \in [\eta_0, \eta_1].$$

Therefore,

$$P^1_{su} u_\eta(\phi)(t) = w_\eta(P_c u_\eta(\phi)(t))$$

and

$$\left|P^1_{su} u_\eta(\phi)(t)\right|_1 < \delta$$

hold for all $\eta \in [\eta_0, \eta_1]$, $\phi \in C_c$ and $t \in \mathbb{R}$. Setting

$$O_\delta = \{\phi \in C^1 : |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\}$$

we obtain

$$\{u_\eta(\phi)(t) : \phi \in C_c,\ t \in \mathbb{R}\} \subset O_\delta, \qquad \eta \in [\eta_0, \eta_1]. \tag{2.12}$$

[4] shows that Theorem A holds by setting

$$C_{c,0} = \{\phi \in C_c : |\phi|_1 < \delta\},\ C^1_{su,0} = \{\phi \in C^1_s \oplus C_u : |\phi|_1 < \delta\},$$
$$N = C_{c,0} + C^1_{su,0} = \{\phi \in C^1 : |\phi|_1 < \delta\},$$
$$w_c = w_{\eta_0}|_{C_{c,0}},\ W_c = \{\phi + w_c(\phi) : \phi \in C_{c,0}\}.$$

It remains to prove that w_c is C^1-smooth, and $Dw_c(0) = 0$.

3 Proof of the C^1-smoothness

Let $j_{\eta_0\eta_1}$ denote the linear continuous inclusion map

$$C^1_{\eta_0} \ni u \mapsto u \in C^1_{\eta_1}.$$

For the fixed point $u_{\eta_0}(\phi)$ of $G_{\eta_0}(\cdot, \phi)$, $\phi \in C_c$, we have

$$\begin{aligned} G_{\eta_1}(j_{\eta_0\eta_1} u_{\eta_0}(\phi), \phi) &= T_e(\cdot)(\phi) + \mathcal{K}(R(u_{\eta_0}(\phi))) \\ &= j_{\eta_0\eta_1} G_{\eta_0}(u_{\eta_0}(\phi), \phi) \\ &= j_{\eta_0\eta_1} u_{\eta_0}(\phi). \end{aligned}$$

So $j_{\eta_0\eta_1} u_{\eta_0}(\phi)$ is a fixed point of $G_{\eta_1}(\cdot, \phi) : C^1_{\eta_1} \to C^1_{\eta_1}$, and by the uniqueness of the fixed point,

$$u_{\eta_1}(\phi) = j_{\eta_0\eta_1} u_{\eta_0}(\phi), \qquad \phi \in C_c.$$

Then $u_{\eta_0}(\phi)(0) = u_{\eta_1}(\phi)(0)$, $\phi \in C_c$, follows, and consequently

$$w_{\eta_0}(\phi) = w_{\eta_1}(\phi), \qquad \phi \in C_c.$$

Let $ev : C^1_{\eta_1} \to C^1$ be given by $ev(u) = u(0)$. Then

$$w_{\eta_1} = P^1_{su} \circ ev \circ u_{\eta_1}.$$

As P^1_{su} and ev are linear bounded maps, in order to conclude the C^1-smoothness of w_{η_1}, it is sufficient to show that $u_{\eta_1} : C_c \to C^1_{\eta_1}$ is C^1-smooth. At the end of the proof we shall conclude $Dw_{\eta_1}(0) = 0$ as well.

We need two abstract results. The first one is a slightly modified version of Lemma II.6 in [7] about the smoothness of substitution operators. Although its proof follows very closely that of Lemma II.6 in [7], we show it here.

Lemma 3.1 *Let E and F be real Banach spaces. For $\eta \geq 0$ set $E_\eta = C_\eta(\mathbb{R}, E)$ and $F_\eta = C_\eta(\mathbb{R}, F)$. Let a subset $O \subset E$ and a continuous and bounded map $q : O \to F$ be given. Consider the substitution operator*

$$Q : O^{\mathbb{R}} \to F^{\mathbb{R}}, \quad Q(u)(t) = q(u(t)) \quad \text{for } u \in O^{\mathbb{R}},\ t \in \mathbb{R}.$$

(i) *If $\eta \geq 0$ and $\tilde\eta \geq 0$, then $Q(O^{\mathbb{R}} \cap E_\eta) \subset F_{\tilde\eta}$.*

(ii) *Assume that O is open, q is C^1-smooth with Dq bounded, and $0 \leq \eta \leq \tilde\eta$. Then, for every $u \in C(\mathbb{R}, O)$, the linear map*

$$A(u) : E^{\mathbb{R}} \to F^{\mathbb{R}}$$

given by

$$A(u)(v)(t) = Dq(u(t))v(t) \qquad \text{for } v \in E^{\mathbb{R}},\ t \in \mathbb{R},$$

satisfies

$$A(u)E_\eta \subset F_{\tilde\eta}, \qquad \sup_{v\in E_\eta,\ |v|_{E_\eta}\le 1} |A(u)v|_{F_{\tilde\eta}} \le \sup_{x\in O} |Dq(x)|,$$

the induced linear maps

$$A_{\eta\tilde\eta}(u) : E_\eta \to F_{\tilde\eta}$$

are continuous, and in case $\eta < \tilde\eta$ *the map*

$$A_{\eta\tilde\eta} : C(\mathbb{R}, O) \cap E_\eta \ni u \mapsto A_{\eta\tilde\eta}(u) \in L(E_\eta, F_{\tilde\eta})$$

is also continuous.

(iii) *If, in addition,* $\eta < \tilde\eta$ *and* O *is convex, then for every* $\tilde\epsilon > 0$ *and* $u \in C(\mathbb{R}, O) \cap E_\eta$ *there exists* $\tilde\delta > 0$ *so that for every* $v \in C(\mathbb{R}, O) \cap E_\eta$ *with* $|v - u|_{E_\eta} < \tilde\delta$, *we have*

$$|Q(v) - Q(u) - A_{\eta\tilde\eta}(u)[v-u]|_{F_{\tilde\eta}} \le \tilde\epsilon |v-u|_{E_\eta}.$$

Proof 1. Proof of (i). The map $\mathbb{R} \ni t \mapsto q(u(t)) \in F$ is continuous provided $u \in O^{\mathbb{R}} \cap E_\eta$, and

$$\sup_{t\in\mathbb{R}} e^{-\tilde\eta|t|}|q(u(t))| \le \sup_{x\in O} |q(x)|,$$

yielding the inclusion.

2. Proof of (ii). For $u \in C(\mathbb{R}, O)$ and $v \in E_\eta$, the map $\mathbb{R} \ni t \mapsto Dq(u(t))v(t) \in F$ is continuous. In addition

$$e^{-\tilde\eta|t|}|Dq(u(t))v(t)| \le e^{-(\tilde\eta-\eta)|t|}e^{-\eta|t|}|v(t)| \sup_{x\in O}|Dq(x)|,$$

yielding $A(u)E_\eta \subset F_{\tilde\eta}$, and

$$\sup_{v\in E_\eta,\ |v|_{E_\eta}\le 1} |A(u)v|_{F_{\tilde\eta}} \le \sup_{x\in O}|Dq(x)|.$$

In order to show the continuity of $A_{\eta\tilde\eta}$, let $u \in C(\mathbb{R}, O) \cap E_\eta$ be given and let $\tilde\epsilon > 0$. Choose $t_0 > 0$ such that

$$2e^{-(\tilde\eta-\eta)|t|} \sup_{x\in O}|Dq(x)| < \tilde\epsilon \qquad \text{for } |t| \ge t_0.$$

There is such a $t_0 > 0$ because of $\tilde\eta > \eta$. Find $\tilde\delta > 0$ so that for all $t \in [-t_0, t_0]$ the ball

$$B_t = \{y \in E : |y - u(t)| < \tilde\delta e^{\eta t_0}\}$$

is contained in O, and for all $y \in B_t$ we have

$$|Dq(y) - Dq(u(t))| < \tilde\epsilon.$$

Consider $\hat u \in C(\mathbb{R}, O) \cap E_\eta$ with $|\hat u - u|_{E_\eta} < \tilde\delta$, and $v \in E_\eta$ with $|v|_{E_\eta} \le 1$. Then, for $|t| \ge t_0$,

$$\begin{aligned} e^{-\tilde\eta|t|}|[Dq(\hat u(t)) - Dq(u(t))]v(t)| &\le 2e^{-(\tilde\eta-\eta)|t|}e^{-\eta|t|}|v(t)| \sup_{x\in O}|Dq(x)| \\ &\le 2e^{-(\tilde\eta-\eta)|t|} \sup_{x\in O}|Dq(x)| \\ &< \tilde\epsilon, \end{aligned}$$

and, for $|t| \le t_0$, by applying $|\hat{u}(t) - u(t)| < \tilde{\delta} e^{\eta|t|} \le \tilde{\delta} e^{\eta t_0}$, we find

$$\begin{aligned} e^{-\tilde{\eta}|t|}|[Dq(\hat{u}(t)) - Dq(u(t))]v(t)| &\le e^{-(\tilde{\eta}-\eta)|t|} e^{-\eta|t|}|v(t)|\,|Dq(\hat{u}(t)) - Dq(u(t))| \\ &\le |Dq(\hat{u}(t)) - Dq(u(t))| \\ &< \tilde{\epsilon}, \end{aligned}$$

It follows that

$$|A_{\eta\tilde{\eta}}(\hat{u}) - A_{\eta\tilde{\eta}}(u)|_{L(E_\eta, F_{\tilde{\eta}})} \le \tilde{\epsilon}.$$

3. Proof of (iii). For all u, v in $C(\mathbb{R}, O) \cap E_\eta$ and $t \in \mathbb{R}$, by using the convexity of O, we obtain

$$\begin{aligned} & e^{-\tilde{\eta}|t|}|q(v(t)) - q(u(t)) - Dq(u(t))[v(t) - u(t)]| \\ &\quad = e^{-\tilde{\eta}|t|}\left|\int_0^1 [Dq(sv(t) + (1-s)u(t)) - Dq(u(t))][v(t) - u(t)]\, ds\right| \\ &\quad \le e^{-(\tilde{\eta}-\eta)|t|} e^{-\eta|t|}|v(t) - u(t)| \max_{s\in[0,1]} |Dq(sv(t) + (1-s)u(t)) - Dq(u(t))| \\ &\quad \le e^{-(\tilde{\eta}-\eta)|t|}|v - u|_{E_\eta} \max_{s\in[0,1]} |Dq(sv(t) + (1-s)u(t)) - Dq(u(t))|. \end{aligned} \tag{3.1}$$

Let $\tilde{\epsilon} > 0$ and $u \in C(\mathbb{R}, O) \cap E_\eta$ be given. Choose $t_0 > 0$ and $\tilde{\delta} > 0$ as in part 2. Let $v \in C(\mathbb{R}, O) \cap E_\eta$ be given with $|v - u|_{E_\eta} < \tilde{\delta}$. For $|t| \ge t_0$, combining (3.1) and the choice of t_0, it follows that

$$\begin{aligned} & e^{-\tilde{\eta}|t|}|q(v(t)) - q(u(t)) - Dq(u(t))[v(t) - u(t)]| \\ & \le 2e^{-(\tilde{\eta}-\eta)|t|} \max_{x\in O} |Dq(x)|\, |v - u|_{E_\eta} \le \tilde{\epsilon}|v - u|_{E_\eta}. \end{aligned}$$

For $|t| \le t_0$, we get $|v(t) - u(t)| \le \tilde{\delta} e^{\eta t_0}$, $|t| \le t_0$. Combining this inequality, the asumption $\eta < \tilde{\eta}$, estimation (3.1), and the choice of $\tilde{\delta}$, one concludes

$$e^{-\tilde{\eta}|t|}|q(v(t)) - q(u(t)) - Dq(u(t))[v(t) - u(t)]| \le \tilde{\epsilon}|v - u|_{E_\eta}.$$

This completes the proof of Lemma 3.1. □

The following C^1-smoothness result is contained in [7]. It is a slightly modified version of that of [8].

Lemma 3.2 *Let X, Λ be Banach spaces over $\mathbb{R}$, and let an open set $P \subset \Lambda$, a map $h : X \times P \to X$ and a constant $\kappa \in [0, 1)$ be given with*

$$|h(x,p) - h(\tilde{x},p)| \le \kappa|x - \tilde{x}|$$

for all $x, \tilde{x}$ in X and all $p \in P$. Consider a convex subset M of X and a map $\Phi : P \to M$ so that for every $p \in P$, $\Phi(p)$ is the unique fixed point of $h(\cdot, p) : X \to X$. Suppose that the following hold.

(i) *The restriction $h_0 = h|_{M\times P}$ has a partial derivative $D_2 h_0 : M \times P \to L(\Lambda, X)$, and $D_2 h_0$ is continuous.*

(ii) *There are a Banach space X_1 over $\mathbb{R}$ and a continuous injective map $j : X \to X_1$ so that the map $k = j \circ h_0$ is continuously differentiable with respect to M in the sense that there is a continuous map*

$$B : M \times P \to L(X, X_1)$$

so that for every $(x,p) \in M \times P$ and every $\epsilon^* > 0$ there exists $\delta^* > 0$ with

$$|k(\tilde{x},p) - k(x,p) - B(x,p)[\tilde{x} - x]| \le \epsilon^*|\tilde{x} - x|$$

for all $\tilde{x} \in M$ *with* $|\tilde{x} - x| \le \delta^*$.

(iii) *There exist maps*

$$h^{(1)} : M \times P \to L(X, X), \quad h_1^{(1)} : M \times P \to L(X_1, X_1)$$

such that

$$B(x,p)\hat{x} = jh^{(1)}(x,p)\hat{x} = h_1^{(1)}(x,p)j\hat{x} \qquad \text{for all } (x,p,\hat{x}) \in M \times P \times X$$

and

$$|h^{(1)}(x,p)| \le \kappa, \ |h_1^{(1)}(x,p)| \le \kappa \qquad \text{for all } (x,p) \in M \times P.$$

(iv) *The map*

$$M \times P \ni (x,p) \mapsto j \circ h^{(1)}(x,p) \in L(X, X_1)$$

is continuous.

Then the map $j \circ \Phi : P \to X_1$ *is* C^1*-smooth, and*

$$D(j \circ \Phi)(p) = h_1^{(1)}(\Phi(p), p) \circ D(j \circ \Phi)(p) + j \circ D_2 h_0(\Phi(p), p) \quad \text{for all } p \in P.$$

Now we employ Lemma 3.2 to prove that the map u_{η_1} is C^1-smooth.

Recall that r_δ restricted to the convex open set O_δ of C^1 is C^1-smooth, and, by inequality (2.9),

$$\sup_{\phi \in O_\delta} |Dr_\delta(\phi)| \le \mu(\delta).$$

An application of Lemma 3.1 with $E = C^1$, $F = Y^{\odot *}$, $O = O_\delta$, $q = l \circ r_\delta$ and $\eta = \eta_0$, $\tilde{\eta} = \eta_1$ shows that the linear maps

$$A(u) : (C^1)^{\mathbb{R}} \to (Y^{\odot *})^{\mathbb{R}}$$

induce a continuous map $A_{\eta_0 \eta_1}$ from the convex set

$$M = \{u \in C^1_{\eta_0} : u(t) \in O_\delta \text{ for all } t \in \mathbb{R}\} \subset C^1_{\eta_0}$$

into $L(C^1_{\eta_0}, Y_{\eta_1})$ so that for every $u \in M$ and for every $\tilde{\epsilon} > 0$ there exists $\tilde{\delta} > 0$ such that for every $v \in M$ with $|v - u|_{C^1_{\eta_0}} \le \tilde{\delta}$, we have

$$R(u) \in Y_{\eta_1}, \ R(v) \in Y_{\eta_1}, \ |R(u) - R(v) - A_{\eta_0\eta_1}(u)[v-u]|_{Y_{\eta_1}} \le \tilde{\epsilon}|v-u|_{C^1_{\eta_0}}. \quad (3.2)$$

Set $X = C^1_{\eta_0}$, $\Lambda = P = C_c$, $h = G_{\eta_0}$, and $\kappa = \frac{1}{2}$. By (2.12), $u_{\eta_0}(P) \subset M$ holds. So, the unique fixed point of $h(\cdot, \phi) : X \to X$ is the map $\Phi : P \to M$ given by $\Phi(\phi) = u_{\eta_0}(\phi)$. In the next steps we verify the hypotheses of Lemma 3.2.

The map $h_0 = h|_{M \times P}$ satisfies

$$h_0(u, \phi) = G_{\eta_0}(u, \phi) = S_{\eta_0}(\phi) + \mathcal{K}_{\eta_0} \circ R_{\eta_0}(u).$$

So, the partial derivative $D_2 h_0 : M \times P \to L(\Lambda, X)$ exists, and for every $(u, \phi) \in M \times P$ it is given by

$$D_2 h_0(u,\phi)\psi = S_{\eta_0}(\psi) \in C^1_{\eta_0}, \qquad \psi \in C_c.$$

It is clearly continuous since it is a constant map.

Setting $X_1 = C^1_{\eta_1}$ and $j = j_{\eta_0\eta_1}$, the map $k = j \circ h_0$ is given by

$$k(u, \phi) = S_{\eta_1}(\phi) + \mathcal{K}_{\eta_1} \circ R_{\eta_1} \circ j(u),$$

and

$$B : M \times P \ni (u, \phi) \mapsto \mathcal{K}_{\eta_1} \circ A_{\eta_0\eta_1}(u) \in L(X, X_1)$$

is continuous.

Let $\epsilon^* > 0$ and $(u, \phi) \in M \times P$ be given. Define

$$\delta^* = \tilde{\delta}\left(\frac{\epsilon^*}{1 + |\mathcal{K}_{\eta_1}|}\right),$$

where $\tilde{\delta}(\tilde{\epsilon})$ is chosen so that (3.2) holds for all $v \in M$ with $|v - u|_{C^1_{\eta_0}} \leq \delta^*$ and $\tilde{\epsilon} = \epsilon^*/(1 + |\mathcal{K}_{\eta_1}|)$. Then for all such $v \in M$ we find

$$\begin{aligned}|k(v,\phi) - & k(u,\phi) - B(u,\phi)[v-u]|_{X_1} \\ &= |\mathcal{K}_{\eta_1}(R(v)) - \mathcal{K}_{\eta_1}(R(u)) - \mathcal{K}_{\eta_1}(A_{\eta_0\eta_1}(u)[v-u])|_{C^1_{\eta_1}} \\ &\leq |\mathcal{K}_{\eta_1}|\frac{\epsilon^*}{1 + |\mathcal{K}_{\eta_1}|}|v-u|_{C^1_{\eta_0}} \\ &\leq \epsilon^*|v-u|_{C^1_{\eta_0}}.\end{aligned}$$

From the facts $\sup_{\phi \in O_\delta} |Dr_\delta(\phi)| \leq \mu(\delta)$ and $|l| = 1$, it is clear that, for every $u \in M$, $A(u)$ defines the elements

$$\mathcal{K}_{\eta_0} \circ A_{\eta_0\eta_0}(u) \in L(X, X) \quad \text{with } |\mathcal{K}_{\eta_0} \circ A_{\eta_0\eta_0}(u)| \leq \mu(\delta)$$

and

$$\mathcal{K}_{\eta_1} \circ A_{\eta_1\eta_1}(u) \in L(X_1, X_1) \quad \text{with } |\mathcal{K}_{\eta_1} \circ A_{\eta_1\eta_1}(u)| \leq \mu(\delta).$$

Now we can define

$$h^{(1)} : M \times P \ni (u, \phi) \mapsto \mathcal{K}_{\eta_0} \circ A_{\eta_0\eta_0}(u) \in L(X, X)$$

and

$$h_1^{(1)} : M \times P \ni (u, \phi) \mapsto \mathcal{K}_{\eta_1} \circ A_{\eta_1\eta_1}(u) \in L(X_1, X_1).$$

For all $(u, \phi, v) \in M \times P \times X$,

$$B(u,\phi)v = \mathcal{K}(A(u)v) = jh^{(1)}(u,\phi)v = h_1^{(1)}(u,\phi)jv$$

holds. Using $|A_{\eta_0\eta_0}(u)| \leq \mu(\delta)$, $|A_{\eta_1\eta_1}(u)| \leq \mu(\delta)$, and the choice of η_0, η_1, δ, it is obvious that

$$|h^{(1)}(u,\phi)| \leq \kappa, \ |h_1^{(1)}(u,\phi)| \leq \kappa \qquad \text{for all } (u,\phi) \in M \times P.$$

For the map $M \times P \ni (x, p) \mapsto j \circ h^{(1)}(x, p) \in L(X, X_1)$ we have

$$j \circ h^{(1)}(u,\phi)v = (j \circ \mathcal{K}_{\eta_0} \circ A_{\eta_0\eta_0}(u))v = \mathcal{K}(A(u)v) = B(u,\phi)v$$

for all $(u, \phi, v) \in M \times P \times X$, and the continuity of $M \times P \ni (x, p) \mapsto j \circ h^{(1)}(x, p) \in L(X, X_1)$ follows from that of B.

Therefore all hypotheses of Lemma 3.2 are fulfilled, and $j \circ \Phi = u_{\eta_1} : C_c \to C^1_{\eta_1}$ is C^1-smooth. Moreover,

$$Du_{\eta_1}(\phi) = h_1^{(1)}(u_{\eta_0}(\phi), \phi) \circ Du_{\eta_1}(\phi) + j \circ D_2h_0(u_{\eta_0}(\phi), \phi)$$

for all $\phi \in C_c$. From $Dr_\delta(0) = 0$, $A(0) = 0$ and $h_1^{(1)}(0,0) = 0$ follows. Then, using also $u_{\eta_0}(0) = 0$, we obtain

$$Du_{\eta_1}(0)\psi = S_{\eta_1}(\psi) \qquad \text{for all } \psi \in C_c.$$

Hence

$$\begin{aligned}Dw_{\eta_1}(0)\psi &= (P^1_{su} \circ ev \circ Du_{\eta_1}(0))\psi \\ &= P^1_{su}\psi = 0\end{aligned}$$

follows for all $\psi \in C_c$. Therefore

$$Dw_{\eta_1}(0) = 0.$$

This completes the proof of Theorem 2.1.

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