Unique Periodic Orbits For Delayed Positive Feedback and the Global Attractor

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Abstract

The delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1))$$

with $\mu > 0$ and a real function f satisfying $f(0) = 0$ and $f' > 0$ models a system governed by delayed positive feedback and instantaneous damping. Recently in [KWW] the geometric, topological and dynamical properties of a 3-dimensional compact invariant set were described in the phase space $C = C([-1, 0], \mathbb{R})$ of initial data for solutions of the equation. In this paper, for a set of μ and f which include examples from neural network theory, we show that this 3-dimensional set is the global attractor, i.e., the compact invariant set which attracts all bounded subsets of C. The proof involves, among others, results on uniqueness and absence of periodic orbits.

Key Words:

Delay differential equation, periodic orbit, uniqueness, global attractor

AMS Subject Classifications:

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1. Introduction

The equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

with a real $\mu > 0$ and a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(0) = 0 \quad \text{and} \quad \xi f(\xi) > 0 \quad \text{for all} \quad \xi \neq 0$$

models a system with a rest point given by the zero $\xi = 0$ of f, which is governed by delayed positive feedback and instantaneous damping. Specific applications occur e.g. in neural network theory, for

$$f(\xi) = \alpha \tanh(\beta\xi)$$

with parameters $\alpha > 0$ and $\beta > 0$ (see e.g. [He], [PMGV] and references therein).

In the monograph [KWW] the geometric, topological and dynamical properties of a 3-dimensional compact invariant set $\overline{W}$ in the state space $C = C([-1, 0], \mathbb{R})$ of initial data for solutions of Eq. (1.1) were investigated, for C^1-functions f which satisfy $f(0) = 0$ and $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, have at least 3 zeros of the characteristic function

$$\mathbb{C} \ni \lambda \mapsto \lambda + \mu - f'(0)e^{-\lambda} \in \mathbb{C}$$

in the open right halfplane, and fulfill weak additional global conditions. It was speculated in [KWW] that for a smaller set of parameters $\mu > 0$ and functions $f : \mathbb{R} \to \mathbb{R}$, the invariant set $\overline{W}$ is in fact the global attractor, i.e., the compact invariant set A which attracts all bounded subsets of C. In the present paper we prove that this is indeed the case, for μ and f which include examples from neural network theory.

As global attractors contain all periodic orbits, and since there is exactly one periodic orbit in $\overline{W}$, the proof of the desired result involves a study of uniqueness of periodic orbits: Based on a technique of Cao [Ca2], which is related to earlier work of Kaplan and Yorke [KY1,KY2], Nussbaum [Nu], and Walther [Wa2], we obtain a first uniqueness result for periodic orbits of the positive feedback equation (1.1).

Among the tools we employ are a discrete Lyapunov functional counting sign changes of elements $\phi \in C \setminus \{0\}$, which goes back to work of Mallet-Paret [MP], and a Poincaré–Bendixson theorem of Mallet-Paret and Sell [MPS2]. Also important are results about Floquet multipliers of periodic solutions [KWW], Poincaré maps, and smooth center-stable manifolds for maps [KWW].

The general reference for any result which is used in the sequel without proof or source is [KWW].

The organization of the paper is as follows: Section 2 contains preliminaries about the semiflow of Eq. (1.1) on the space C, the existence of a global attractor, a precise statement of the main result of [KWW], the definition and basic properties of the discrete Lyapunov functional, and the hypotheses on μ and f we use. Section 3 is devoted to results on uniqueness and absence of periodic orbits. In Section 4, the equation $A = \overline{W}$ is proved. In an Appendix we carry out the proof of a simple result which relates solutions of Eq. (1.1) converging to a periodic orbit for $t \to -\infty$ to backward trajectories of a Poincaré map, and which is familiar in case of ordinary differential equations.

Notation, preliminaries. $\mathbb{N}$ and $\mathbb{R}^+$ stand for the nonnegative integers and reals, respectively. S^1 and $S^1_{\mathbb{C}}$ denote the unit circles in $\mathbb{R}^2$ and $\mathbb{C}$, respectively. Simple closed curves are either injective continuous maps from $S^1_{\mathbb{C}}$ into $\mathbb{R}^n$, or continuous maps c from a compact interval $[a,b] \subset \mathbb{R}$, $a < b$, into $\mathbb{R}^n$ so that $c|_{[a,b)}$ is injective and $c(a) = c(b)$. The set of values of a simple closed curve c, or trace, is denoted by $|c|$. The Jordan curve theorem guarantees that the complement of the trace of a simple closed curve c in $\mathbb{R}^2$ consists of two nonempty connected open sets, one bounded and the other unbounded, and $|c|$ is the boundary of each of these components. We denote the bounded component by $\mathrm{int}(c)$ and the unbounded one by $\mathrm{ext}(c)$.

The interior and the boundary of a subset M of a topological space are denoted by $\overset{\circ}{M}$ and ∂M, respectively.

For a Banach space E and $r > 0$ we set

$$E_r = \{x \in E : ||x|| < r\}.$$

Spectra of continuous linear maps $T : E \to E$ are defined as spectra of their complexifications. If a decomposition

$$E = F \oplus G$$

into closed linear subspaces is given then $\mathrm{Pr}_F : E \to E$ and $\mathrm{Pr}_G : E \to E$ denote the associated projection operators along G onto F and along F onto G, respectively.

For a given continuous function $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $\tau > 0$, solutions $x : \mathbb{R} \to \mathbb{R}$ of the equation

$$\dot{x}(t) = g(t, x(t), x(t-\tau)) \tag{1.2}$$

are differentiable functions which satisfy Eq. (1.2) everywhere. If $I \subset \mathbb{R}$ is an interval and if $t_0 \in I$, $\tau > 0$ are given with $t_0 - \tau = \min I$ and $t_0 < \sup I \le \infty$, and if a continuous function $g : (I \cap [t_0, \infty)) \times \mathbb{R}^2 \to \mathbb{R}$ is given, then a continuous function $x : I \to \mathbb{R}$ is a solution of Eq. (1.2) if x is differentiable on $I \cap (t_0, \infty)$ and satisfies Eq. (1.2) for all $t \in I \cap (t_0, \infty)$. It is then clear how to define complex-valued solutions of equations given by functions of the form

$$g(t, x, y) = a(t)x + b(t)y.$$

For a parameter $\tau > 0$, a map $x : D \to M$, and $t \in \mathbb{R}$ so that $[t-\tau, t] \subset D$ the segment $x_t : [-\tau, 0] \to M$ is defined by $x_t(s) = x(t+s)$ for $-\tau \le s \le 0$.

C denotes the Banach space of continuous functions $\phi : [-1, 0] \to \mathbb{R}$, with the norm given by

$$||\phi|| = \max_{-1 \le t \le 0} |\phi(t)|.$$

C^1 is the Banach space of all C^1-maps $\phi : [-1, 0] \to \mathbb{R}$, with the norm given by

$$||\phi||_1 = ||\phi|| + ||\dot{\phi}||.$$

2. The global attractor, the structure of an invariant set, and a discrete Lyapunov functional

We begin with solutions of the positive feedback equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

where

(H0) $\mu > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, with $f(0) = 0$ and $|f(\xi)| < \mu|\xi|$ for all ξ outside a bounded neighbourhood of 0.

The particular case

$$f : \mathbb{R} \ni \xi \mapsto \alpha \tanh(\beta\xi) \in \mathbb{R} \qquad \text{with } \alpha > 0 \text{ and } \beta > 0 \tag{2.1}$$

is used in neural network theory.

We need to recall some basic facts. Every $\phi \in C$ uniquely determines a solution $x^\phi : [-1, \infty) \to \mathbb{R}$ with $x_0^\phi = \phi$. Any two solutions on a common domain are equal whenever they coincide on an interval of length one. The set of values of constant solutions coincides with the zeroset of $f - \mu\,\mathrm{id}$. We have continuous dependence on initial data in the sense that given $\phi \in C$, $t \ge 0$, $\epsilon > 0$ there exists $\delta > 0$ so that $|x^\psi(s) - x^\phi(s)| < \epsilon$ for all $s \in [-1, t]$ and all $\psi \in C$ with $||\psi - \phi|| < \delta$.

The map

$$F : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

is a continuous semiflow. Its stationary points are the constant functions in C given by the zeros of $f - \mu\,\mathrm{id}$. All maps $F(t, \cdot) : C \to C$, $t \ge 0$, are injective and monotone with respect to the pointwise ordering on C given by the cone

$$K = \{\phi \in C : \phi(s) \ge 0 \text{ for all } s \in [-1, 0]\}.$$

All maps $F(t, \cdot)$, $t \ge 1$, are compact (i.e., send bounded sets into sets with compact closure), and all maps

$$C \ni \phi \mapsto F(t, \phi) \in C^1, \qquad t \ge 1,$$

are continuous. Concerning boundedness properties, we have the following result.

Proposition 2.1. *Let $r > 0$ be given with $|f(\xi)| < \mu|\xi|$ for $|\xi| \geq r$. Then*

$$F(\mathbb{R}^+ \times C_r) \subset C_r,$$

and for every $\phi \in C$ there exists $t \geq 0$ so that

$$F(s,\phi) \in C_r \qquad \text{for all } s \geq t.$$

Proof. 1. We show
$$\sup_{t \geq -1} |x(t)| \leq \max\{r, ||x_0||\}$$
for every solution $x : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1). Otherwise there exist a solution $x : [-1,\infty) \to \mathbb{R}$, $\delta > 0$, and $s > 0$ so that $|x(s)| > \delta + \max\{r, ||x_0||\}$. There is a minimal $t > 0$ with $|x(t)| = \delta + \max\{r, ||x_0||\}$. Suppose $x(t) > 0$. Then $\dot{x}(t) \geq 0$. By the minimality of t, $|x(t-1)| < \delta + \max\{r, ||x_0||\}$. The last inequality, the monotonicity of f, the inequality $|f(\xi)| < \mu|\xi|$ for $|\xi| \geq r$, and Eq. (1.1) combined yield $\dot{x}(t) < 0$, which is a contradiction. The argument in case $x(t) < 0$ is analogous.

2. It follows that $F(\mathbb{R}^+ \times C_r) \subset C_r$. Let $\phi \in C$. Set $x = x^\phi$. According to part 1, x is bounded. Let
$$c = \limsup_{t \to \infty} |x(t)|.$$
In case $c < r$ we obtain $t \geq 0$ with $||F(s,\phi)|| = ||x_s|| < r$ for all $s \geq t$.

Suppose $c \geq r$. We have $c = \limsup_{t\to\infty} x(t)$ or $c = -\liminf_{t\to\infty} x(t)$. Consider the first case. There exist $\delta > 0$ and $T \geq 0$ such that

$$-\mu(c-\delta) + f(c+\delta) < -\delta$$

and
$$x(s) \leq c + \delta \qquad \text{for all } s \geq T.$$

We have $x(t) < c - \delta$ for some $t \geq T + 1$ since otherwise

$$\dot{x}(s) \leq -\mu(c-\delta) + f(c+\delta) \qquad \text{for all } s \geq T+1,$$

which yields a contradiction. It follows that $x(s) < c - \delta$ for all $s \geq t$ since otherwise there exists $u > t$ with $x(s) < c - \delta$ for $t \leq s < u$, $x(u) = c - \delta$, and $\dot{x}(u) \geq 0$, in contradiction to
$$\dot{x}(u) \leq -\mu(c-\delta) + f(c+\delta) < -\delta < 0.$$

The last result yields
$$\limsup_{s\to\infty} x(s) \leq c - \delta,$$

which is a contradiction. The argument in the second case is analogous. □

Using the Arzèla–Ascoli theorem, Eq. (1.1) and boundedness of solutions on $[-1,\infty)$ we now obtain that for every $\phi \in C$ the ω-limit set

$$\omega(\phi) = \{\psi \in C : \text{There exists a sequence } (t_n)_0^\infty \text{ in } \mathbb{R}^+ \text{ with } t_n \to \infty \text{ and } F(t_n,\phi) \to \psi \text{ as } n \to \infty\}$$

is nonempty. ω-limit sets are compact, connected, and invariant in the sense that for every $\psi \in \omega(\phi)$ there is a solution $x : \mathbb{R} \to \mathbb{R}$ with $x_0 = \psi$ and $x_t \in \omega(\phi)$ for all $t \in \mathbb{R}$. For bounded solutions $x : \mathbb{R} \to \mathbb{R}$, the α-limit set

$$\alpha(x) = \{\psi \in C : \text{There exists a sequence } (t_n)_0^\infty \text{ in } \mathbb{R} \text{ with } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected, and invariant.

Proposition 2.1 and arguments as in Chapter 17 of [KWW], or in [Ha], yield the existence of a global attractor of the semiflow F, i.e., of a nonempty compact set $A \subset C$ which is invariant in the sense that

$$F(t,A) = A \qquad \text{for all } t \geq 0$$

and which attracts bounded sets in the sense that for every bounded set $B \subset C$ and for every open set $U \supset A$ there exists $t \geq 0$ with

$$F([t,\infty) \times B) \subset U.$$

Global attractors are uniquely determined.

The compactness of A, its invariance property and the injectivity of the maps $F(t,\cdot)$, $t \geq 0$, combined permit to show that the map

$$[0,\infty) \times A \ni (t,\phi) \mapsto F(t,\phi) \in A$$

extends to a continuous flow

$$F_A : \mathbb{R} \times A \to A;$$

for every $\phi \in A$ and for all $t \in \mathbb{R}$ we have

$$F_A(t,\phi) = x_t$$

with the uniquely determined solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) satisfying $x_0 = \phi$. In particular, A is contained in the set

$$B = \{\phi \in C : \text{There are a bounded solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1) and } t \in \mathbb{R} \text{ so that } \phi = x_t\}.$$

Proposition 2.2. *$A = B$.*

Proof. Proof of $B \subseteq A$: Every $\phi \in B$ is contained in an orbit $X = \{x_t : t \in \mathbb{R}\}$ of a bounded solution $x : \mathbb{R} \to \mathbb{R}$. We have $F(t, X) = X$ for all $t \geq 0$. Then the attraction property of A shows that the bounded set X is contained in every open set $U \supset A$. Hence $\mathrm{dist}(\phi, A) = 0$, and by the closedness of A, $\phi \in A$.

The compactness and invariance properties of A yield that A is contained in the set B. □

Note that we have

$$A = F(1, A) \subset C^1;$$

A is a closed subset of C^1. Using the flow F_A and the continuity of the map

$$C \ni \phi \mapsto F(1, \phi) \in C^1$$

one obtains that C and C^1 define the same topology on A.

The aim of the present paper is to show that for subsets of parameters μ and functions f with property (H0), which include examples of the form (2.1), the global attractor coincides with an invariant set whose fine structure we know very well from the results in [KWW]. Next, we introduce this invariant set, for $\mu > 0$ and C^1-functions $f : \mathbb{R} \to \mathbb{R}$ so that

(H1) $f(0) = 0$, $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, $f - \mu\,\mathrm{id}$ has exactly one zero ξ^- in $(-\infty, 0)$ and exactly one zero ξ^+ in $(0, \infty)$, $f'(\xi^-) < \mu$, $f'(\xi^+) < \mu$,

and

(H2) $f'(0) > \frac{\mu}{\cos\theta}$ for $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$ with $\theta = -\mu \tan\theta$.

These hypotheses are stronger than those used in [KWW]. Observe that (H1) implies (H0).

We begin with linearizing the semiflow F at its stationary point 0. The smoothness of f implies that each map $F(t, \cdot)$, $t \geq 0$, is continuously differentiable. For all ϕ, ψ in C and $t \geq 0$ we have

$$D_2F(t, \phi)\psi = v_t$$

with the solution $v : [-1, \infty) \to \mathbb{R}$ of the linear variational equation

$$\dot{v}(s) = -\mu v(s) + f'(x^\phi(s-1))v(s-1)$$

along x^ϕ which is given by $v_0 = \psi$. The operators

$$D_2F(t, 0), \quad t \geq 0,$$

form a strongly continuous semigroup; for $\phi = 0$ the linear variational equation is

$$\dot{v}(t) = -\mu v(t) + f'(0)v(t-1). \tag{2.2}$$

The spectrum σ of the generator of the semigroup $(D_2F(t,0))_{t\geq 0}$ consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \mu - f'(0)e^{-\lambda} = 0 \tag{2.3}$$

obtained from an Ansatz with exponential functions for solutions of Eq. (2.2). Due to $f'(0) > \frac{\mu}{\cos\theta} > \mu$, there is exactly one positive $\lambda_0 \in \sigma$. The remaining points in σ are given by a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_1^\infty$ with

$$\lambda_0 > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \dots, \qquad 2j\pi - \pi < \operatorname{Im}\lambda_j < 2j\pi$$

for $1 \leq j \in \mathbb{N}$, and $\operatorname{Re}\lambda_j \to -\infty$ as $j \to \infty$. Every $\lambda \in \sigma$ is a simple eigenvalue, i.e., has a 1-dimensional generalized eigenspace. The inequality in condition (H2) is equivalent to

$$\operatorname{Re}\lambda_1 > 0.$$

Let P, L, Q denote the realified generalized eigenspaces of the spectral sets $\{\lambda_0\}$, $\{\lambda_1, \overline{\lambda_1}\}$, and $\sigma \setminus \{\lambda_0, \lambda_1, \overline{\lambda_1}\}$, respectively. Then

$$C = P \oplus L \oplus Q,\ \dim P = 1,\ P = \mathbb{R}\eta_0 \text{ with } \eta_0 \in \overset{\circ}{K},\ \dim L = 2,$$

and there exists a 3-dimensional C^1-submanifold W_{loc} of C which is locally positively invariant under F and has tangent space $P \oplus L$ at $0 \in W_{loc}$; for every $\phi \in W_{loc}$ there exists a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that $x_t \in W_{loc}$ for t in some unbounded interval $I \subset (-\infty, 0)$, and $x_t \to 0$ as $t \to -\infty$. Clearly W_{loc} and its forward extension $W = F(\mathbb{R}^+ \times W_{loc})$ are subsets of the unstable set

$$\begin{aligned} W^u(0) = \{\phi \in C :& \text{There is a solution } x : \mathbb{R} \to \mathbb{R} \text{ of Eq. (1.1)} \\ & \text{with } x_0 = \phi \text{ and } x_t \to 0 \text{ as } t \to -\infty\} \end{aligned}$$

of the stationary point 0, which is contained in A, according to Propositions 2.2 and 2.1. The closure $\overline{W} \subseteq A$ is the set which will turn out to be equal to A (and to $\overline{W^u(0)}$), for a subset of the pairs (μ, f) with properties (H1) and (H2).

Before describing the structure of $\overline{W}$ in detail we address a more elementary result on solution behaviour related to monotonicity. Solutions $x : [-1, \infty) \to \mathbb{R}$ which are strictly positive (negative) on an interval $[t-1, t]$, $t \geq 0$, satisfy $x_s \in \overset{\circ}{K}$ $(\in -\overset{\circ}{K})$ for all $s \geq t$; the remaining solutions are those with all segments x_t in the positively invariant set

$$S = \{\phi \in C :\ (x^\phi)^{-1}(0) \text{ is unbounded}\}.$$

The set S is a Lipschitz graph over a closed hyperplane, more precisely, there exists a Lipschitz continuous map $\mathrm{Sep} : L \oplus Q \to P$ so that

$$S = \{\chi + \mathrm{Sep}(\chi) : \chi \in L \oplus Q\}.$$

For ϕ above S, i.e., $\mathrm{Pr}_P\, \phi - \mathrm{Sep}(\mathrm{Pr}_{L\oplus Q}\, \phi) = a\eta_0$ with $a > 0$, there exists $t \geq 0$ with $F(s, \phi) \in \overset{\circ}{K}$ for all $s \geq t$, while for ϕ below S, $F([t, \infty) \times \{\phi\}) \subset -\overset{\circ}{K}$ for some $t \geq 0$. So the graph S is a separatrix for nonoscillatory solution behaviour. For every $\phi \neq \psi$ in S,

$$\phi - \psi \in C \setminus (K \cup (-K)),$$

i.e., no pair of different points in S is in order relation.

We return to the set $\overline{W}$. Observe that condition (H1) implies that F has exactly 3 stationary points, namely the constant functions $\xi_- \in C$ with value ξ^-, 0, and $\xi_+ \in C$ with value ξ^+; and that ξ_- and ξ_+ are stable and hyperbolic.

Theorem 2.3.

(i) $\overline{W}$ *is compact and contains* $\xi_-, 0, \xi_+$. *For every* $\phi \in \overline{W}$,

$$\xi_- \leq \phi \leq \xi_+.$$

For every $\phi \in \overline{W}$ *there exists a uniquely determined solution* $x(\phi) : \mathbb{R} \to \mathbb{R}$ *of Eq. (1.1) with* $x(\phi)_0 = \phi$, *and* $x(\phi)_t \in \overline{W}$ *for all* $t \in \mathbb{R}$. *The map*

$$F_W : \mathbb{R} \times \overline{W} \ni (t, \phi) \mapsto x(\phi)_t \in \overline{W}$$

is a continuous flow. W *and* $\mathrm{bd}\, W = \overline{W} \setminus W$ *are invariant under* F_W. *For every* $\phi \in W$, $\alpha(x(\phi)) = \{0\}$. *The points* $\phi \in \overline{W} \setminus S$ *above (below)* S *form a connected set and satisfy* $F_W(t, \phi) \to \xi_+$ $(\to \xi_-)$ *as* $t \to \infty$. *The set* $\mathcal{O} = \overline{W} \cap S \setminus (W \cap S)$ *is the orbit in* C *of a nonconstant periodic solution* $p : \mathbb{R} \to \mathbb{R}$ *of Eq. (1.1) with minimal period* $\omega \in (1, 2)$. *For every* $\phi \in \mathrm{bd}\, W \setminus \{\xi_-, \xi_+\}$, $\alpha(x(\phi)) = \mathcal{O}$, *and for every* $\phi \in W \cap S \setminus \{0\}$, $\omega(\phi) = \mathcal{O}$.

(ii) There are subspaces G_1, G_2, G_3 *of* C *with dimensions* $1, 2, 3$, *respectively, and a closed subspace* E *of* C *with*

$$C = G_3 \oplus E, \quad G_3 = G_2 \oplus G_1,$$

and compact subsets

$$D_W \quad \textit{of} \quad G_3 \quad \textit{with} \quad D_W = \partial D_W \cup \overset{\circ}{D}_W$$

and

$$D_S \quad \text{of} \quad G_2 \quad \text{with} \quad D_S = \partial D_S \cup \overset{\circ}{D}_S,$$

and continuous maps

$$w : D_W \to E, \quad w_S : D_S \to G_1 \oplus E$$

so that

$$\begin{aligned} \overline{W} &= \{\chi + w(\chi) : \chi \in D_W\}, \\ W &= \{\chi + w(\chi) : \chi \in \overset{\circ}{D}_W\}, \\ \overline{W} \cap S &= \{\chi + w_S(\chi) : \chi \in D_S\}, \\ \mathcal{O} &= \{\chi + w_S(\chi) : \chi \in \partial D_S\}. \end{aligned}$$

The set ∂D_S is the trace of a simple closed C^1-curve c, and $\overset{\circ}{D}_S$ is the bounded component of $G_2 \setminus |c|$. The map w_S is C^1-smooth in the sense that its restriction to $\overset{\circ}{D}_S$ is C^1-smooth and that for every $\chi \in \partial D_S$ there is an open neighbourhood N of χ in G_2 so that $w_S|_{N \cap D_S}$ extends to a C^1-map on N. The complement in ∂D_W of the set $\{\chi_-, \chi_+\}$ given by

$$\xi_- = \chi_- + w(\chi_-), \quad \xi_+ = \chi_+ + w(\chi_+),$$

is a 2-dimensional C^1-submanifold of G_3, and the restriction of w to $D_W \setminus \{\chi_-, \chi_+\}$ is C^1-smooth in the sense that the restriction of w to $\overset{\circ}{D}_W$ is C^1-smooth and for every $\chi \in \partial D_W \setminus \{\chi_-, \chi_+\}$ there is an open neighbourhood N of χ in G_3 so that $w|N \cap D_W$ extends to a C^1-map on N. There exist homeomorphisms from $\overline{W}$ and D_W onto the closed unit ball in $\mathbb{R}^3$ which send $\operatorname{bd} W$ and ∂D_W onto the unit sphere.

One may visualize the invariant set $\overline{W}$ as a smooth solid spindle which is split by the invariant disk $\overline{W} \cap S$ into the basins of attraction towards the tips ξ_- and ξ_+.

Observe that $\mathcal{O} = \overline{W} \cap S \setminus (W \cap S)$ is the only periodic orbit in $\overline{W}$, and recall that according to Proposition 2.2 the global attractor A contains all periodic orbits. So, a necessary condition for the desired equation

$$A = \overline{W}$$

is that the semiflow F has exactly one periodic orbit. This uniqueness property is violated for many μ and f satisfying (H1) and (H2). Simple examples are given by $\mu > 0$ and f linear in a neighbourhood of 0 so that $\operatorname{Re} \lambda_k = 0$ for some integer $k \geq 2$ — in these cases there exist periodic solutions

$$t \mapsto a \cos((\operatorname{Im} \lambda_k) t), \quad a \neq 0,$$

with minimal period $\frac{2\pi}{\operatorname{Im}\lambda_k} < 1 < \omega$. We shall consider $\mu > 0$ and C^1-functions $f : \mathbb{R} \to \mathbb{R}$ so that in addition to (H1) and (H2) the following properties hold:

(H3) $f'(0) < \frac{\mu}{\cos\theta}$ for $\theta \in (4\pi - \frac{\pi}{2}, 4\pi)$ with $\theta = -\mu \tan\theta$,

(H4) $f(\xi) = -f(-\xi)$ for all $\xi \in \mathbb{R}$, and
the function $(0, \infty) \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$ is strictly decreasing.

Condition (H3) is equivalent to $\operatorname{Re}\lambda_2 < 0$, as we shall see. All functions f of the form (2.1) satisfy condition (H4); for suitable μ, α, β also (H1), (H2), (H3) are satisfied.

The first major steps in the proof of

$$A \subseteq \overline{W}$$

are to exclude periodic orbits different from $\mathcal{O}$, and then to deduce from a Poincaré–Bendixson theorem [MPS2] that as α-limit sets of solutions on $\mathbb{R}$ with segments in A only the subsets

$$\{\xi_-\},\ \{0\},\ \{\xi_+\},\ \text{and } \mathcal{O} \text{ of } \overline{W}$$

occur.

At the end of this section we recall the definition and properties of a discrete Lyapunov functional

$$V : C \setminus \{0\} \to 2\mathbb{N} \cup \{\infty\}$$

which goes back to work of Mallet-Paret [MP] and older observations of Myshkis [My] and others that for certain delay differential equations the oscillation frequency of solutions does not increase with t. The functional played an important role in [KWW] and will also be used in the sequel.

The definition is as follows, First, set

$$\begin{aligned} \operatorname{sc}(\phi) = \sup\big\{ & k \in \mathbb{N} \setminus \{0\} : \text{There is a strictly increasing finite sequence} \\ & (s^i)_0^k \text{ in } [-1, 0] \text{ with } \phi(s^{i-1})\phi(s^i) < 0 \text{ for all } i \in \{1, 2, \ldots, k\}\big\} \le \infty \end{aligned}$$

for $\phi \in C \setminus (K \cup (-K))$, and $\operatorname{sc}(\phi) = 0$ for $0 \neq \phi \in K \cup (-K)$. Then, define

$$V(\phi) = \begin{cases} \operatorname{sc}(\phi) & \text{if } \operatorname{sc}(\phi) \in 2\mathbb{N} \cup \{\infty\}, \\ \operatorname{sc}(\phi) + 1 & \text{if } \operatorname{sc}(\phi) \in 2\mathbb{N} + 1. \end{cases}$$

Set

$$\begin{aligned} R = \{\phi \in C^1 : & \phi(0) \neq 0 \text{ or } \dot{\phi}(0)\phi(-1) > 0, \\ & \phi(-1) \neq 0 \text{ or } \dot{\phi}(-1)\phi(0) < 0, \\ & \text{all zeros of } \phi \text{ in } (-1, 0) \text{ are simple}\}. \end{aligned}$$

The next proposition lists basic properties of V.

Proposition 2.4.

(i) *For every* $\phi \in C \setminus \{0\}$ *and for every sequence* $(\phi_n)_0^\infty$ *in* $C \setminus \{0\}$ *with* $\phi_n \to \phi$ *as* $n \to \infty$,

$$V(\phi) \leq \liminf_{n\to\infty} V(\phi_n).$$

(ii) *For every* $\phi \in R$ *and for every sequence* $(\phi_n)_0^\infty$ *in* $C^1 \setminus \{0\}$ *with* $||\phi_n - \phi||_{C^1} \to 0$ *as* $n \to \infty$,

$$V(\phi) = \lim_{n\to\infty} V(\phi_n) < \infty.$$

(iii) *Let an interval* $I \subset \mathbb{R}$, *a real* $\nu \geq 0$, *and continuous functions* $b : I \to (0, \infty)$ *and* $z : I + [-1, 0] \to \mathbb{R}$ *be given so that* $z|_I$ *is differentiable with*

$$\dot{z}(t) = -\nu z(t) + b(t)z(t-1) \tag{2.4}$$

for $\inf I < t \in I$, *and* $z(t) \neq 0$ *for some* $t \in I + [-1, 0]$. *Then the map* $I \ni t \mapsto V(z_t) \in 2\mathbb{N} \cup \{\infty\}$ *is decreasing, with*

$$V(z_t) = \infty \text{ or } V(z_{t-2}) > V(z_t) \qquad \text{in case } t \in I,\ t-2 \in I,\ z(t) = 0 = z(t-1).$$

For all $t \in I$ *with* $t - 3 \in I$ *and* $V(z_{t-3}) = V(z_t)$, *we have* $z_t \in R$.

Observe that linear variational equations

$$\dot{v}(t) = -\mu v(t) + f'(x(t-1))v(t-1)$$

along solutions of Eq. (1.1) are of the form considered in the last statement, as well as the equation satisfied by weighted differences $y = \frac{1}{c}(x - \hat{x})$, $c \neq 0$, of solutions $x, \hat{x}$ of Eq. (1.1) on a common domain,

$$\dot{y}(t) = -\mu y(t) + \left(\int_0^1 f'((1-s)\hat{x}(t-1) + sx(t-1))\, ds\right) y(t-1).$$

The next a-priori estimate is a special case of a result which says that solutions with finite oscillation frequency do not decay too fast as t increases. Estimates of this type go back to Walther [Wa1] and Mallet-Paret [MP], see also [Ar] and [Ca1].

Proposition 2.5. *For every* $\nu > 0$, b_0, *and* $b_1 \geq b_0$ *there is* $k > 0$ *so that for every* $t_0 \in \mathbb{R}$, *and for every continuous function* $b : [t_0 - 5, t_0] \to \mathbb{R}$ *with range in* $[b_0, b_1]$, *and for every solution* $z : [t_0 - 6, t_0] \to \mathbb{R}$ *of Eq. (2.4) with* $z_{t_0-5} \neq 0$ *and* $V(z_{t_0-5}) \leq 2$, *we have*

$$||z_{t_0-1}|| \leq k||z_{t_0}||.$$

Finally we recall that the periodic solution p of Eq. (1.1) satisfies

$$p_t \in R \text{ and } V(p_t) = 2 = V(\dot{p}_t) \qquad \text{for all } t \in \mathbb{R},$$

$$p_t - p_s \in R \text{ and } V(p_t - p_s) = 2 \qquad \text{for all } t \neq s \text{ in } [0, \omega);$$

and that it was normalized in [KWW] so that $p(0) = 0$ and $\dot{p}(0) > 0$.

3. Periodic orbits

Throughout this section we assume that $\mu > 0$ and the C^1-function $f : \mathbb{R} \to \mathbb{R}$ satisfy (H1) and (H4). Conditions (H2) and (H3) are supposed only if stated explicitly. We obtain results on uniqueness and on absence of periodic orbits with prescribed oscillation frequencies, i.e., in given level sets of V. Our approach uses the technique of Cao [Ca2] who studied slowly oscillating periodic solutions of negative feedback equations, which have zeros spaced at distances larger than the delay. The periodic solutions of the equations considered here have higher oscillation frequencies, and not all arguments from [Ca2] can easily be adapted to the present situation. We overcome these difficulties by means of the oddness condition in (H4).

It is convenient to consider a further equation,

$$\dot{z}(t) = -\mu z(t) + g(z(t - \alpha)) \tag{3.1}$$

with $\mu > 0$, $\alpha > 0$, and a C^1-function $g : \mathbb{R} \to \mathbb{R}$ which is odd ($g(\xi) = -g(-\xi)$ for all $\xi \in \mathbb{R}$) and satisfies

$$g'(\xi) > 0 \qquad \text{for all } \xi \in \mathbb{R}.$$

If z is a solution of Eq. (3.1) then the function w given by $w(t) = z(\alpha t)$ is a solution of the equation

$$\dot{w}(t) = \alpha \left[-\mu w(t) + g(w(t - 1)) \right]. \tag{3.2}$$

We begin with a-priori information on periodic solutions of Eq. (3.2) which follows from general results in [MPS2] for certain systems of delay differential equations.

Proposition 3.1. *Let $w : \mathbb{R} \to \mathbb{R}$ be a nonconstant periodic solution of Eq. (3.2) with minimal period $T_w > 0$.*

(i) There are $t^0 \in \mathbb{R}$ and $t^1 \in (t^0, t^0 + T_w)$ with $0 < \dot{w}(t)$ for $t^0 < t < t^1$, $w(\mathbb{R}) = [w(t^0), w(t^1)]$, $\dot{w}(t) < 0$ for $t^1 < t < t^0 + T_w$, and

$$c_w : [0, T_w] \ni t \mapsto \begin{pmatrix} w(t) \\ w(t-1) \end{pmatrix} \in \mathbb{R}^2$$

is a simple closed curve.

(ii) *If w has a zero then*

$$w\left(t+\frac{T_w}{2}\right)=-w(t) \qquad \text{for all } t\in\mathbb{R}.$$

(iii) $\{w_t : t\in\mathbb{R}\}\subset V^{-1}(2k)$ *for some* $k\in\mathbb{N}$.

(iv) *For every nonconstant periodic solution $y:\mathbb{R}\to\mathbb{R}$ of Eq. (3.2) with minimal period $T_y>0$ and $y_t\neq w_s$ for all t,s in $\mathbb{R}$, we have $|c_y|\cap|c_w|=\emptyset$.*

We have $w(t^0)\neq 0\neq w(t^1)$ in assertion (i), as follows from assertion (iii) in combination with the last statement in Proposition 2.4 and the definition of R.

Observe that for every solution $z:\mathbb{R}\to\mathbb{R}$ of Eq. (3.1) and for the solution $w:\mathbb{R}\ni t\mapsto z(\alpha t)\in\mathbb{R}$ of Eq. (3.2) we have

$$\left\{\begin{pmatrix} z(t)\\ \dot z(t)\end{pmatrix} : t\in\mathbb{R}\right\}=\left\{\begin{pmatrix} w(t)\\ \frac{1}{\alpha}\dot w(t)\end{pmatrix} : t\in\mathbb{R}\right\}.$$

Notice that for every solution $w:\mathbb{R}\to\mathbb{R}$ of Eq. (3.2) and for every $t\in\mathbb{R}$, the values $w(t)$ and $\frac{1}{\alpha}\dot w(t)$ uniquely determine $w(t-1)$. Proposition 3.1 combined with these facts yields the following result.

Corollary 3.2. *Let $z:\mathbb{R}\to\mathbb{R}$ be a nonconstant periodic solution of Eq. (3.1) with minimal period $T_z>0$. The map*

$$Z:[0,T_z]\ni t\mapsto\begin{pmatrix} z(t)\\ \dot z(t)\end{pmatrix}\in\mathbb{R}^2$$

is a simple closed curve. If z has a zero then

$$0\in\operatorname{int}(Z).$$

Let $x:\mathbb{R}\to\mathbb{R}$ be a nonconstant periodic solution of Eq. (3.1) with minimal period $T_x>0$. Suppose that the functions $w:\mathbb{R}\ni t\mapsto z(\alpha t)\in\mathbb{R}$ and $y:\mathbb{R}\ni t\mapsto x(\alpha t)\in\mathbb{R}$ satisfy

$$y_t\neq w_s \qquad \text{for all } t,s \text{ in } \mathbb{R}.$$

Then the traces of Z and of the simple closed curve

$$X:[0,T_x]\ni t\mapsto\begin{pmatrix} x(t)\\ \dot x(t)\end{pmatrix}\in\mathbb{R}^2$$

are disjoint.

For $\theta\in[0,2\pi)$ define

$$l(\theta)=\{r(\cos\theta,\sin\theta)^{tr}\in\mathbb{R}^2 : r\geq 0\}.$$

Proposition 3.3. *Let $z : \mathbb{R} \to \mathbb{R}$ be a nonconstant periodic solution of Eq. (3.1) with minimal period $T > 0$ and with a zero, and let a maximum $a \in \mathbb{R}$ of z be given. Then the functions*

$$\psi : [0, 2\pi) \ni \theta \mapsto \inf\{t \in (a, a+T] : (z(t), \dot{z}(t))^{tr} \in l(\theta)\} \in \mathbb{R}$$

and

$$\Psi : [0, 2\pi) \ni \theta \mapsto \sup\{t \in (a, a+T] : (z(t), \dot{z}(t))^{tr} \in l(\theta)\} \in \mathbb{R}$$

are strictly decreasing.

Proof. Proposition 3.1(i) yields the existence of reals a_1, a_2, a_3 so that $a < a_1 < a_2 < a_3 < a + T$ and

$$\begin{aligned}
&z(a_1) = \dot{z}(a_2) = z(a_3) = 0, \\
&z(t) > 0 \qquad \text{for } a < t < a_1 \text{ and for } a_3 < t \leq a + T, \\
&\dot{z}(t) < 0 \qquad \text{for } a < t < a_2, \\
&z(t) < 0 \qquad \text{for } a_1 < t < a_3, \\
&\dot{z}(t) > 0 \qquad \text{for } a_2 < t < a + T.
\end{aligned}$$

Clearly, $\psi(0) = a + T$, $\psi(\pi/2) = a_3$ and

$$\psi(\theta) \in (a_3, a+T) \qquad \text{for all } \theta \in \left(0, \frac{\pi}{2}\right).$$

Let $\theta \in (0, \pi/2)$ be given. Observe that

$$(z(t), \dot{z}(t))^{tr} \in l(\theta) \qquad \text{for some } t \in (a, a+T]$$

if and only if

$$t \in (a_3, a+T) \quad \text{and} \quad \theta = \arctan \frac{\dot{z}(t)}{z(t)}.$$

The function

$$(a_3, a+T) \ni t \mapsto \arctan \frac{\dot{z}(t)}{z(t)} \in \mathbb{R}$$

is continuous with range in $(0, \pi/2)$, and

$$\lim_{t \to a_3^+} \arctan \frac{\dot{z}(t)}{z(t)} = \frac{\pi}{2}, \qquad \lim_{t \to a+T^-} \arctan \frac{\dot{z}(t)}{z(t)} = 0.$$

It follows that

$$\psi(\theta) = \min\left\{t \in (a_3, a+T) : \theta = \arctan \frac{\dot{z}(t)}{z(t)}\right\}.$$

Let θ_1 and θ_2 be given in $(0, \pi/2)$ with $\theta_1 < \theta_2$. Suppose

$$\psi(\theta_1) \leq \psi(\theta_2).$$

Then there exist t_1 and t_2 in $(a_3, a+T)$ with

$$t_i = \psi(\theta_i), \quad \theta_i = \arctan \frac{\dot{z}(t_i)}{z(t_i)}, \quad i \in \{1, 2\}.$$

From $t_1 = \psi(\theta_1) \leq \psi(\theta_2) = t_2$ and $\theta_1 < \theta_2$ we infer

$$t_1 < t_2.$$

The facts

$$\lim_{t \to a_3^+} \arctan \frac{\dot{z}(t)}{z(t)} = \frac{\pi}{2}$$

and

$$0 < \arctan \frac{\dot{z}(t_1)}{z(t_1)} < \arctan \frac{\dot{z}(t_2)}{z(t_2)} < \frac{\pi}{2}$$

yield a real $t_3 \in (a_3, t_1)$ so that

$$\arctan \frac{\dot{z}(t_3)}{z(t_3)} = \arctan \frac{\dot{z}(t_2)}{z(t_2)} = \theta_2.$$

This contradicts the definition of $\psi(\theta_2)$ and t_2. Consequently, $\psi(\theta_1) > \psi(\theta_2)$, and hence ψ is strictly decreasing on $[0, \pi/2]$.

The proofs for the intervals $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi)$ and for the function Ψ are analogous. $\square$

The next result is the key to uniqueness and absence of periodic orbits. It is analogous to an earlier result of Cao [Ca2] for slowly oscillating periodic solutions.

Proposition 3.4. *Let $k \in \mathbb{N} \setminus \{0\}$, $\alpha \geq 1$. Let an odd C^1-function $g : \mathbb{R} \to \mathbb{R}$ be given which satisfies $g'(0) = f'(0)$, and*

$$g(\xi) > f(\xi) \quad \text{and} \quad \frac{g'(\xi)}{g(\xi)} > \frac{f'(\xi)}{f(\xi)} \qquad \text{for all } \xi > 0.$$

Let a nonconstant periodic solution x of Eq. (1.1) with minimal period $T_x > 0$ and a nonconstant periodic solution z of Eq. (3.1) with minimal period $T_z > 0$ be given. Suppose $w : \mathbb{R} \ni t \mapsto z(\alpha t) \in \mathbb{R}$ satisfies $V(x_t) = V(w_t) = 2k$ for all $t \in \mathbb{R}$. Then for

$$X : [0, T_x] \ni t \mapsto \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathbb{R}^2 \quad \text{and} \quad Z : [0, T_z] \ni t \mapsto \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \in \mathbb{R}^2$$

the following situation cannot occur:

$$\begin{gathered}|Z| \subset |X| \cup \operatorname{ext}(X), \\ |Z| \cap |X| \neq \emptyset, \\ r|Z| \subset \operatorname{ext}(X) \qquad \textit{for all } r > 1.\end{gathered}$$

Proof. Assume that for the closed curves X and Z,

$$\begin{gathered}|Z| \subset |X| \cup \operatorname{ext}(X), \\ |Z| \cap |X| \neq \emptyset, \\ r|Z| \subset \operatorname{ext}(X) \qquad \text{for all } r > 1\end{gathered}$$

hold. Our aim is to get a contradiction.

The assumptions $k \in \mathbb{N} \setminus \{0\}$ and $V(x_t) = V(w_t) = 2k$ for all $t \in \mathbb{R}$ yield $x(t) = 0$ for some $t \in \mathbb{R}$ and $z(s) = 0$ for some $s \in \mathbb{R}$. Corollary 3.2 implies $0 \in \operatorname{int}(X)$ and $0 \in \operatorname{int}(Z)$.

Observe that, for every $\theta \in [0, 2\pi)$, any point of $l(\theta) \cap |Z|$ is not closer to $0 \in \mathbb{R}^2$ than any point of $l(\theta) \cap |X|$.

Using $|Z| \cap |X| \neq \emptyset$, we may assume $X(0) = Z(0)$ without loss of generality, i.e.,

$$x(0) = z(0) \quad \text{and} \quad \dot{x}(0) = \dot{z}(0). \tag{3.3}$$

We distinguish two cases.

Case 1: $\dot{x}(0) = \dot{z}(0) = 0$.

Then $c = x(0) = z(0) \neq 0$ since $0 \in \operatorname{int}(X)$ and $0 \in \operatorname{int}(Z)$. Assume $c > 0$ (The proof in case $c < 0$ is analogous). Proposition 3.1(i) and (ii) yield that x and z have the special symmetry $x(t + T_x/2) = -x(t)$ and $z(t + T_z/2) = -z(t)$ for all $t \in \mathbb{R}$, and

$$\begin{gathered}c = \max_{t\in\mathbb{R}} x(t) = \max_{t\in\mathbb{R}} z(t), \\ -c = \min_{t\in\mathbb{R}} x(t) = \min_{t\in\mathbb{R}} z(t), \\ \dot{x}(t) > 0 \qquad \text{for } -\frac{T_x}{2} < t < 0, \\ \dot{z}(t) > 0 \qquad \text{for } -\frac{T_z}{2} < t < 0, \\ x(-\frac{T_x}{2}) = -c, \quad \dot{x}(-\frac{T_x}{2}) = 0, \\ z(-\frac{T_z}{2}) = -c, \quad \dot{z}(-\frac{T_z}{2}) = 0.\end{gathered}$$

Let $T^* = \min\{T_x, T_z\}$.

Claim: $z(s) \le x(s)$ for $-T^*/2 \le s \le 0$.

Proof of the Claim: Let x^{-1} and z^{-1} denote the inverses of the functions

$$[-\frac{T_x}{2}, 0] \ni t \mapsto x(t) \in \mathbb{R}$$

and

$$[-\frac{T_z}{2}, 0] \ni t \mapsto z(t) \in \mathbb{R},$$

respectively. Then the domain of x^{-1} and z^{-1} is $[-c, c]$. The functions

$$\phi_x : [-c, c] \ni u \mapsto \dot{x}(x^{-1}(u)) \in \mathbb{R},$$

$$\phi_z : [-c, c] \ni u \mapsto \dot{z}(z^{-1}(u)) \in \mathbb{R}$$

satisfy $\phi_x(-c) = \phi_x(c) = \phi_z(-c) = \phi_z(c) = 0$, and $\phi_x(u) > 0$, $\phi_z(u) > 0$ for all $u \in (-c, c)$. The arcs

$$\Omega_x = \left\{X(t) : t \in [-\frac{T_x}{2}, 0]\right\} \quad \text{and} \quad \Omega_z = \left\{Z(t) : t \in [-\frac{T_z}{2}, 0]\right\}$$

coincide with the graphs

$$\left\{(u, \phi_x(u))^{tr} : u \in [-c, c]\right\} \quad \text{and} \quad \left\{(u, \phi_z(u))^{tr} : u \in [-c, c]\right\},$$

respectively. From the special symmetry of x and z we obtain

$$|X| = \Omega_x \cup (-\Omega_x) \quad \text{and} \quad |Z| = \Omega_z \cup (-\Omega_z).$$

Hence

$$\mathrm{int}(X) = \left\{(u, v)^{tr} : u \in (-c, c),\ -\phi_x(-u) < v < \phi_x(u)\right\}.$$

From $|Z| \subset |X| \cup \mathrm{ext}(X)$ we conclude

$$\phi_x(u) \le \phi_z(u) \qquad \text{for } -c \le u \le c.$$

The functions x and z satisfy

$$\dot{x}(t) = \phi_x(x(t)) \qquad \text{for all } t \in [-\frac{T_x}{2}, 0]$$

and

$$\dot{z}(t) = \phi_z(z(t)) \qquad \text{for all } t \in [-\frac{T_z}{2}, 0].$$

For $-T_z/2 < s_1 < s_2 < 0$ the last equation and the inequality $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$ combined yield

$$\int_{z(s_1)}^{z(s_2)} \frac{du}{\phi_z(u)} = \int_{s_1}^{s_2} \frac{\dot{z}(t)}{\phi_z(z(t))}\, dt = s_2 - s_1.$$

Also,

$$\int_{x(s_1)}^{x(s_2)} \frac{du}{\phi_x(u)} = s_2 - s_1 \qquad \text{for } \frac{-T_x}{2} < s_1 < s_2 < 0.$$

The continuity of z and x at 0 yields

$$\int_{z(s)}^{c} \frac{du}{\phi_z(u)} = -s \qquad \text{for } \frac{-T_z}{2} < s \leq 0$$

and

$$\int_{x(s)}^{c} \frac{du}{\phi_x(u)} = -s \qquad \text{for } \frac{-T_x}{2} < s \leq 0.$$

For $-T^*/2 < s \leq 0$ we obtain immediately that for every such s,

$$\int_{z(s)}^{c} \frac{du}{\phi_z(u)} = \int_{x(s)}^{c} \frac{du}{\phi_x(u)},$$

and using $0 < \phi_x(u) \leq \phi_z(u)$ on $(-c, c)$ we infer

$$z(s) \leq x(s) \qquad \text{for } \frac{-T^*}{2} < s \leq 0.$$

Using continuity we complete the proof of the claim.

If $T_z > T_x$ then from the claim above and from $x(-T_x/2) = -c$ we obtain $z(-T_x/2) \leq x(-T_x/2) = -c$. This is impossible since $-T_z/2 < -T_x/2 < 0$, $z(-T_z/2) = -c$ and $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$. So

$$T_z \leq T_x.$$

Proposition 3.1 implies that all zeros of z are simple and the distance between two consecutive zeros of z is $T_z/2$. This fact and the equations $V(w_t) = 2k$ for all $t \in \mathbb{R}$ combined yield

$$(k - \frac{1}{2})T_z \leq \alpha \leq kT_z.$$

Using $V(w_t) = 2k$ for all $t \in \mathbb{R}$ and the last assertion of Proposition 2.4(iii) we infer

$$(k - \frac{1}{2})T_z < \alpha < kT_z.$$

One can similarly obtain

$$(k-\frac{1}{2})T_x < 1 < kT_x.$$

Combining $\alpha \geq 1$, $T_z \leq T_x$ and the last two inequalities we get

$$(k-\frac{1}{2})T_z \leq (k-\frac{1}{2})T_x < 1 \leq \alpha < kT_z \leq kT_x. \tag{3.4}$$

Hence

$$-kT_z < -\alpha \leq -1 < -(k-\frac{1}{2})T_z \tag{3.5}$$

follows. Using the inequalities $\dot{z}(t) > 0$ for $-T_z/2 < t < 0$, Proposition 3.1(i), and the periodicity of z, we conclude that

$$\dot{z}(t) < 0 \qquad \text{for } -kT_z < t < -(k-\frac{1}{2})T_z.$$

This inequality and (3.5) combined imply

$$z(-\alpha) \geq z(-1).$$

From the periodicity and the special symmetry of z we find

$$z(-1) = z(-1+kT_z) = -z(-1+(k-\frac{1}{2})T_z).$$

From (3.4) we obtain

$$-\frac{T_z}{2} < -1+(k-\frac{1}{2})T_z < 0.$$

These inequalities, the claim above and $T^* = T_z$ combined yield

$$z(-1+(k-\frac{1}{2})T_z) \leq x(-1+(k-\frac{1}{2})T_z).$$

Using that x is increasing on $[-\frac{T_x}{2}, 0]$ and the consequence

$$-\frac{T_x}{2} \leq -\frac{T_z}{2} < -1+(k-\frac{1}{2})T_z \leq -1+(k-\frac{1}{2})T_x < 0$$

of the inequalities $T_z \leq T_x$ and (3.4), we infer

$$x(-1+(k-\frac{1}{2})T_z) \leq x(-1+(k-\frac{1}{2})T_x).$$

By the periodicity and the special symmetry of x,

$$x(-1+(k-\frac{1}{2})T_x) = -x(-1+kT_x) = -x(-1).$$

Consequently,

$$\begin{aligned} z(-\alpha) &\geq z(-1) \\ &= -z(-1+(k-\frac{1}{2})T_z) \\ &\geq -x(-1+(k-\frac{1}{2})T_z) \\ &\geq -x(-1+(k-\frac{1}{2})T_x) \\ &= x(-1). \end{aligned} \tag{3.6}$$

Using Eq. (3.1) and Eq. (1.1) and $\dot{z}(0) = \dot{x}(0) = 0$, $z(0) = x(0) = c > 0$ we obtain

$$z(-\alpha) > 0, \quad x(-1) > 0$$

and

$$g(z(-\alpha)) = f(x(-1)).$$

Using the inequalities $g(\xi) > f(\xi)$ for all $\xi > 0$ we infer

$$f(x(-1)) = g(z(-\alpha)) > f(z(-\alpha)).$$

The last inequality and the monotonicity of f yield

$$z(-\alpha) < x(-1),$$

which contradicts (3.6).

Case 2: $\dot{z}(0) = \dot{x}(0) \neq 0$.

Let $d = z(0) = x(0)$. There exists $\epsilon > 0$ such that $\dot{z}(t) \neq 0$ and $\dot{x}(t) \neq 0$ for all $t \in (-\epsilon, \epsilon)$. Then there is $\delta > 0$ so that there are inverses

$$z^{-1} : (d-\delta, d+\delta) \to \mathbb{R}, \quad x^{-1} : (d-\delta, d+\delta) \to \mathbb{R}$$

of restrictions of z and x to open intervals in $(-\epsilon, \epsilon)$, respectively. The maps

$$\begin{aligned} \eta_z &: (d-\delta, d+\delta) \ni u \mapsto \dot{z}(z^{-1}(u)) \in \mathbb{R}, \\ \eta_x &: (d-\delta, d+\delta) \ni u \mapsto \dot{x}(x^{-1}(u)) \in \mathbb{R} \end{aligned}$$

are C^1-smooth since, from equations (1.1) and (3.1), x and z are C^2-smooth. We have $\eta_z(d) = \dot{z}(0) = \dot{x}(0) = \eta_x(d) \neq 0$, and for all $u \in (d-\delta, d+\delta)$,

$$\begin{aligned} {\eta_x}'(u) &= \ddot{x}(x^{-1}(u))\frac{d}{du}x^{-1}(u) = \ddot{x}(x^{-1}(u))\frac{1}{\dot{x}(x^{-1}(u))}, \\ {\eta_z}'(u) &= \ddot{z}(z^{-1}(u))\frac{d}{du}z^{-1}(u) = \ddot{z}(z^{-1}(u))\frac{1}{\dot{z}(z^{-1}(u))}. \end{aligned}$$

In particular,

$$\eta_x{}'(d) = \frac{\ddot{x}(0)}{\dot{x}(0)} \quad \text{and} \quad \eta_z{}'(d) = \frac{\ddot{z}(0)}{\dot{z}(0)}. \tag{3.7}$$

The sets

$$\left\{(u, \eta_x(u))^{tr} : u \in (d-\delta, d+\delta)\right\}, \quad \left\{(u, \eta_z(u))^{tr} : u \in (d-\delta, d+\delta)\right\}$$

are graph representations of pieces of $|X|$ and $|Z|$, respectively. It is not difficult to show that there exists $\gamma > 0$ so that the sets

$$\left\{(u, v)^{tr} : u \in (d - \frac{\delta}{2}, d + \frac{\delta}{2}),\ \eta_x(u) - \gamma < v < \eta_x(u)\right\}$$

and

$$\left\{(u, v)^{tr} : u \in (d - \frac{\delta}{2}, d + \frac{\delta}{2}),\ \eta_x(u) < v < \eta_x(u) + \gamma\right\}$$

belong to different connected components of $\mathbb{R}^2 \setminus |X|$. Hence, using $|Z| \subset |X| \cup \mathrm{ext}(X)$ and $(d, \eta_z(d)) = (d, \eta_x(d))$, we infer

$$\eta_z{}'(d) = \eta_x{}'(d).$$

This fact, the relations $\dot{z}(0) = \dot{x}(0) \neq 0$, and (3.7) combined imply

$$\ddot{z}(0) = \ddot{x}(0).$$

Differentiation of Eq. (1.1) and of Eq. (3.1) yields

$$\begin{aligned}
\ddot{x}(t) &= -\mu\dot{x}(t) + f'(x(t-1))\dot{x}(t-1), \\
\ddot{z}(t) &= -\mu\dot{z}(t) + g'(z(t-\alpha))\dot{z}(t-\alpha).
\end{aligned}$$

From $\ddot{z}(0) = \ddot{x}(0)$ and $\dot{z}(0) = \dot{x}(0)$ we obtain

$$g'(z(-\alpha))\dot{z}(-\alpha) = f'(x(-1))\dot{x}(-1). \tag{3.8}$$

Eq. (1.1) and Eq. (3.1) at $t = 0$, and (3.3) combined yield

$$g(z(-\alpha)) = f(x(-1)).$$

There are three subcases of Case 2:

$$\begin{aligned}
\text{Either } & z(-\alpha) > 0,\ x(-1) > 0, \\
\text{or } & z(-\alpha) < 0,\ x(-1) < 0, \\
\text{or } & z(-\alpha) = x(-1) = 0.
\end{aligned}$$

Case 2.1: $z(-\alpha) > 0$, $x(-1) > 0$.

The equality $g(z(-\alpha)) = f(x(-1))$ and the inequality $g(\xi) > f(\xi)$ for all $\xi > 0$ combined imply

$$f(x(-1)) = g(z(-\alpha)) > f(z(-\alpha)).$$

Hence, by the monotonicity of f,

$$0 < z(-\alpha) < x(-1).$$

From

$$\frac{g'(\xi)}{g(\xi)} > \frac{f'(\xi)}{f(\xi)} \qquad \text{for all } \xi > 0$$

we find

$$\xi\frac{g'(\xi)}{g(\xi)} > \xi\frac{f'(\xi)}{f(\xi)} \qquad \text{for all } \xi > 0.$$

This fact, $0 < z(-\alpha) < x(-1)$, and (H4) combined yield

$$z(-\alpha)\frac{g'(z(-\alpha))}{g(z(-\alpha))} > z(-\alpha)\frac{f'(z(-\alpha))}{f(z(-\alpha))} > x(-1)\frac{f'(x(-1))}{f(x(-1))} > 0. \tag{3.9}$$

The equation (3.8) and $g(z(-\alpha)) = f(x(-1))$ and the inequality $0 < z(-\alpha) < x(-1)$ combined yield

$$z(-\alpha)\frac{g'(z(-\alpha))}{g(z(-\alpha))}\frac{\dot{z}(-\alpha)}{z(-\alpha)} = x(-1)\frac{f'(x(-1))}{f(x(-1))}\frac{\dot{x}(-1)}{x(-1)}. \tag{3.10}$$

According to (3.9) and (3.10) we can distinguish three subcases of Case 2.1:

$$\begin{aligned} \text{Either } & \dot{z}(-\alpha) = \dot{x}(-1) = 0, \\ \text{or } & 0 < \frac{\dot{z}(-\alpha)}{z(-\alpha)} < \frac{\dot{x}(-1)}{x(-1)}, \\ \text{or } & \frac{\dot{x}(-1)}{x(-1)} < \frac{\dot{z}(-\alpha)}{z(-\alpha)} < 0. \end{aligned}$$

Case 2.1.1: $\dot{z}(-\alpha) = \dot{x}(-1) = 0$.

We have $Z(-\alpha) \notin \mathrm{int}(X)$, and

$$Z(-\alpha) \in \{(u,0)^{tr} \in \mathbb{R}^2 : u > 0\}.$$

It is easy to see from Proposition 3.1 that

$$\{(u,0)^{tr} \in \mathbb{R}^2 : 0 \le u < x(-1)\} \subset \mathrm{int}(X).$$

Consequently,

$$z(-\alpha) \geq x(-1),$$

which contradicts $z(-\alpha) < x(-1)$.

Case 2.1.2: $0 < \frac{\dot{z}(-\alpha)}{z(-\alpha)} < \frac{\dot{x}(-1)}{x(-1)}$.

Choose $a \in \mathbb{R}$ so that $z(a) = \max_{t \in \mathbb{R}} z(t)$ and $-\alpha \in (a, a+T_z]$. Select the reals a_1, a_2, a_3 so that $a < a_1 < a_2 < a_3 < a + T_z$ and

$$\begin{aligned}
&z(a_1) = \dot{z}(a_2) = z(a_3) = 0, \\
&z(t) > 0 \qquad \text{for } a < t < a_1 \text{ and for } a_3 < t \leq a + T_z, \\
&\dot{z}(t) < 0 \qquad \text{for } a < t < a_2, \\
&z(t) < 0 \qquad \text{for } a_1 < t < a_3, \\
&\dot{z}(t) > 0 \qquad \text{for } a_2 < t < a + T_z.
\end{aligned}$$

Recall Proposition 3.3. Define

$$\theta_z = \arctan \frac{\dot{z}(-\alpha)}{z(-\alpha)} \quad \text{and} \quad \theta_x = \arctan \frac{\dot{x}(-1)}{x(-1)}.$$

Then θ_z, θ_x are in $(0, \pi/2)$, $\theta_z < \theta_x$ and

$$Z(-\alpha) \in l(\theta_z), \quad X(-1) \in l(\theta_x).$$

Let $t_* = \psi(\theta_x)$. Observe $t_* \in [a_3, a + T_z)$. Recall that, for every $\theta \in [0, 2\pi)$, any point of $|Z| \cap l(\theta)$ is not closer to the origin than any point of $|X| \cap l(\theta)$. Consequently,

$$z(t_*) \geq x(-1) > 0.$$

The monotonicity of ψ and the fact $Z(-\alpha) \in |Z| \cap l(\theta_z)$ combined yield

$$-\alpha \geq \psi(\theta_z) > \psi(\theta_x) = t_*.$$

As z is strictly increasing on $[a_3, a + T_z]$, and $a_3 \leq t_* < -\alpha \leq a + T_z$ we infer

$$z(t_*) < z(-\alpha).$$

The last inequality and $z(t_*) \geq x(-1) > 0$ together imply

$$z(-\alpha) > x(-1),$$

a contradiction to $z(-\alpha) < x(-1)$.

Case 2.1.3: $\frac{\dot{x}(-1)}{x(-1)} < \frac{\dot{z}(-\alpha)}{z(-\alpha)} < 0$.

Define

$$\theta_z = \arctan \frac{\dot{z}(-\alpha)}{z(-\alpha)} + 2\pi \quad \text{and} \quad \theta_x = \arctan \frac{\dot{x}(-1)}{x(-1)} + 2\pi.$$

Then θ_z, θ_x are in $(3\pi/2, 2\pi)$, $\theta_x < \theta_z$ and

$$Z(-\alpha) \in l(\theta_z), \quad X(-1) \in l(\theta_x).$$

Choose a, a_1, a_2, a_3 as in Case 2.1.2, and apply Proposition 3.3 to z as in Case 2.1.2. Let $t^* = \Psi(\theta_x)$. Observe $t^* \in (a, a_1]$. Analogously to Case 2.1.2, we find

$$z(t^*) \geq x(-1) > 0.$$

Proposition 3.3 and $Z(-\alpha) \in |Z| \cap l(\theta_z)$ combined yield

$$t^* = \Psi(\theta_x) > \Psi(\theta_z) \geq -\alpha.$$

Now we use the inequality $a < -\alpha < t^* \leq a_1$ and the fact that z is strictly decreasing on $(a, a_1]$ to obtain

$$z(-\alpha) > z(t^*).$$

Then we arrive at the contradiction

$$z(-\alpha) > x(-1).$$

Case 2.2: $z(-\alpha) < 0$, $x(-1) < 0$.

Arguments analogous to those used in Case 2.1 lead to a contradiction.

Case 2.3: $z(-\alpha) = x(-1) = 0$.

Then equality (3.8) and the assumption $g'(0) = f'(0) \neq 0$ combined imply

$$\dot{z}(-\alpha) = \dot{x}(-1).$$

Here $\dot{z}(-\alpha) = \dot{x}(-1) \neq 0$ since $0 \in \text{int}(Z)$ and $0 \in \text{int}(X)$.

From Eq. (3.1) at $t = -\alpha$ and from Eq. (1.1) at $t = -1$, we obtain

$$z(-2\alpha) \neq 0, \quad x(-2) \neq 0.$$

Defining $\tilde{z} : \mathbb{R} \ni t \mapsto z(t - \alpha) \in \mathbb{R}$ and $\tilde{x} : \mathbb{R} \ni t \mapsto x(t-1) \in \mathbb{R}$, and replacing z and x with $\tilde{z}$ and $\tilde{x}$, respectively, either Case 2.1 or Case 2.2 holds. As both cases lead to contradictions we arrive at a contradiction also in Case 2.3. $\square$

In order to introduce quantities which are explicit in μ and $f'(0)$ and characterize uniqueness and absence of periodic solutions of Eq. (1.1) we make a digression and consider the characteristic equations

$$\zeta + \alpha[\mu - f'(0)e^{-\zeta}] = 0 \tag{3.11}$$

obtained from an Ansatz $t \mapsto e^{\zeta t}$ for complex-valued solutions of the linear equations

$$\dot{y}(t) = \alpha\,[-\mu y(t) + f'(0) y(t-1)] \tag{3.12}$$

with parameter $\alpha > 0$. The results for Eq. (2.3) are applicable and yield functions $\zeta_0 : (0,\infty) \to \mathbb{R}$ and $\zeta_j : (0,\infty) \to \mathbb{R} + i(2j\pi - \pi, 2j\pi)$, $0 \neq j \in \mathbb{N}$, so that for each $\alpha > 0$ the solutions of Eq. (3.11) are

$$\zeta_0(\alpha), \zeta_1(\alpha), \overline{\zeta_1(\alpha)}, \dots ,$$

and

$$\zeta_0(\alpha) > \operatorname{Re}\zeta_1(\alpha) > \dots , \quad \operatorname{Re}\zeta_j(\alpha) \to -\infty \quad \text{as } j \to -\infty.$$

All functions ζ_j, $j \in \mathbb{N}$, are analytic. Obviously, the solutions of Eq. (2.3) are given by

$$\lambda_j = \zeta_j(1).$$

From $f'(0) > \mu$ in (H1) we get

$$\zeta_0(\alpha) > 0 \qquad \text{for every } \alpha > 0.$$

If $0 \neq j \in \mathbb{N}$ then there is a uniquely determined parameter α_j so that

$$\operatorname{Re}\zeta_j(\alpha_j) = 0;$$

we have

$$\operatorname{Im}\zeta_j(\alpha_j) = 2j\pi - \arccos\frac{\mu}{f'(0)}$$

and

$$\alpha_j = \frac{2j\pi - \arccos\frac{\mu}{f'(0)}}{\sqrt{f'(0)^2 - \mu^2}}.$$

Clearly,

$$\alpha_j < \alpha_{j+1} \qquad \text{for } 0 \neq j \in \mathbb{N}.$$

We compute

$$(\operatorname{Re}\zeta_j)'(\alpha_j) > 0 \qquad \text{for } 0 \neq j \in \mathbb{N}$$

and conclude that for every integer $j > 0$ the inequality

$$\alpha_j \geq 1 \quad (> 1)$$

is equivalent to $\operatorname{Re}\zeta_j(1) \leq 0$ (< 0), or

$$\operatorname{Re}\lambda_j \leq 0 \quad (< 0)$$

in terms of the eigenvalues of the generator of the semigroup $(D_2F(t,0))_{t\geq 0}$.

Recall from Proposition 3.1 that for any nonconstant periodic solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) there exists $k\in\mathbb{N}$ so that $V(x_t)=2k$ for all $t\in\mathbb{R}$. For $k\in\mathbb{N}$, we say that Eq. (1.1) has a periodic orbit in $V^{-1}(2k)$ if it has a nonconstant periodic solution $x:\mathbb{R}\to\mathbb{R}$ with $V(x_t)=2k$ for all $t\in\mathbb{R}$.

The main result of this section is summarized in the following theorem.

Theorem 3.5. *Assume that hypotheses (H1) and (H4) are satisfied.*

(i) For every $k\in\mathbb{N}\setminus\{0\}$, Eq. (1.1) has at most one periodic orbit in $V^{-1}(2k)$.

(ii) For every $k\in\{0\}\cup\{j\in\mathbb{N}\setminus\{0\}:\alpha_j\geq 1\}$, Eq. (1.1) has no periodic orbit in $V^{-1}(2k)$.

Proof. 1. Suppose that $k\in\mathbb{N}\setminus\{0\}$ and there exist nonconstant periodic solutions $x:\mathbb{R}\to\mathbb{R}$ and $y:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) with minimal periods $T_x>0$ and $T_y>0$, respectively, so that $V(x_t)=V(y_t)=2k$ for all $t\in\mathbb{R}$, and $\{x_t:t\in[0,T_x]\}\cap\{y_t:t\in[0,T_y]\}=\emptyset$. Consider

$$X:[0,T_x]\ni t\mapsto\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\in\mathbb{R}^2\quad\text{and}\quad Y:[0,T_y]\ni t\mapsto\begin{pmatrix}y(t)\\ \dot{y}(t)\end{pmatrix}\in\mathbb{R}^2.$$

Corollary 3.2 implies that either $|X|\subset\operatorname{int}(Y)$ or $|Y|\subset\operatorname{int}(X)$. Without loss of generality assume $|Y|\subset\operatorname{int}(X)$. Corollary 3.2 implies also $0\in\operatorname{int}(X)$ and $0\in\operatorname{int}(Y)$. It follows that $\rho|Y|\subset\operatorname{ext}(X)$ for all sufficiently large $\rho>0$. Define

$$\beta=\inf\{\rho\geq 0:\rho'|Y|\subset\operatorname{ext}(X)\text{ for all }\rho'>\rho\}.$$

Then $\beta>1$, $\rho|Y|\subset\operatorname{ext}(X)$ for all $\rho>\beta$, $\beta|Y|\cap|X|\neq\emptyset$, and

$$\beta|Y|\subset|X|\cup\operatorname{ext}(X).$$

The function

$$g:\mathbb{R}\ni\xi\mapsto\beta f\left(\frac{\xi}{\beta}\right)\in\mathbb{R}$$

is odd and continuously differentiable. The function

$$z:\mathbb{R}\ni t\mapsto\beta y(t)\in\mathbb{R}$$

is a T_y-periodic solution of Eq. (3.1) with $\alpha=1$ and g as just defined. Clearly, $V(z_t)=2k$ for all $t\in\mathbb{R}$. Setting

$$Z:[0,T_y]\ni t\mapsto\begin{pmatrix}z(t)\\ \dot{z}(t)\end{pmatrix}\in\mathbb{R}^2,$$

we have

$$|Z| \subset |X| \cup \operatorname{ext}(X),$$

$$|Z| \cap |X| \neq \emptyset,$$

$$r|Z| \subset \operatorname{ext}(X) \qquad \text{for all } r > 1.$$

Using $\lim_{\xi \to 0} \frac{\xi f'(\xi)}{f(\xi)} = 1$ and hypothesis (H4) we infer

$$\frac{\xi f'(\xi)}{f(\xi)} < 1 \qquad \text{for all } \xi > 0.$$

For every $\xi > 0$, the function

$$(0, \infty) \ni u \mapsto u f\left(\frac{\xi}{u}\right) \in \mathbb{R}$$

is strictly increasing since its derivatives at $u > 0$ are given by

$$f\left(\frac{\xi}{u}\right)\left[1 - \frac{\frac{\xi}{u} f'\left(\frac{\xi}{u}\right)}{f\left(\frac{\xi}{u}\right)}\right] > 0.$$

This fact and $\beta > 1$ combined imply

$$g(\xi) = \beta f\left(\frac{\xi}{\beta}\right) > f(\xi) \qquad \text{for all } \xi > 0.$$

Using (H4) and $\beta > 1$ we obtain

$$\frac{g'(\xi)}{g(\xi)} = \frac{1}{\xi} \frac{\frac{\xi}{\beta} f'\left(\frac{\xi}{\beta}\right)}{f\left(\frac{\xi}{\beta}\right)} > \frac{1}{\xi} \frac{\xi f'(\xi)}{f(\xi)} = \frac{f'(\xi)}{f(\xi)}$$

for all $\xi > 0$. Clearly, $g'(0) = f'(0)$.

Now Proposition 3.4 yields a contradiction. This proves assertion (i).

2. We show that Eq. (1.1) has no periodic orbit in $V^{-1}(0)$. Suppose the contrary, i.e., that $x : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq. (1.1) with $V(x_t) = 0$ for all $t \in \mathbb{R}$. The last statement in Proposition 2.4, the definition of R and periodicity of x combined yield $x(t) \neq 0$ for all $t \in \mathbb{R}$. Assume $x(t) > 0$ for all $t \in \mathbb{R}$ (The proof for the case $x(t) < 0$ for all $t \in \mathbb{R}$ is analogous). Set $m = \min_{t \in \mathbb{R}} x(t)$ and $M = \max_{t \in \mathbb{R}} x(t)$. As x is nonconstant, either $m \in (0, \xi^+)$ or $M > \xi^+$ holds. If $m \in (0, \xi^+)$ and $s \in \mathbb{R}$ is given

so that $x(s) = m$ then $\dot{x}(s) = 0$. From hypothesis (H1) we obtain $-\mu\xi + f(\xi) > 0$ for $0 < \xi < \xi^+$. Using this and the monotonicity of f we get

$$\dot{x}(s) = -\mu x(s) + f(x(s-1)) \geq -\mu x(s) + f(x(s)) > 0,$$

which is a contradiction.

If $M > \xi^+$ and $u \in \mathbb{R}$ is chosen so that $x(u) = M$ then $\dot{x}(u) = 0$. On the other hand, the inequality $-\mu\xi + f(\xi) < 0$ for $\xi > \xi^+$ and the monotonicity of f combined yield

$$\dot{x}(u) = -\mu x(u) + f(x(u-1)) \leq -\mu x(u) + f(x(u)) < 0,$$

a contradiction.

3. Let $k \in \mathbb{N} \setminus \{0\}$ be given with $\alpha_k \geq 1$. Suppose that $x : \mathbb{R} \to \mathbb{R}$ is a nonconstant periodic solution of Eq. (1.1) with minimal period $T_x > 0$ and with $V(x_t) = 2k$ for all $t \in \mathbb{R}$.

The remarks about the characteristic equations (3.11) imply that

$$w^k : \mathbb{R} \ni t \mapsto \cos\left((\operatorname{Im}\zeta_k(\alpha_k))t\right) \in \mathbb{R}$$

is a nontrivial periodic solution of Eq. (3.12) with $\alpha = \alpha_k$. We have $V(w^k_t) = 2k$ for all $t \in \mathbb{R}$. It follows that

$$y^k : \mathbb{R} \ni t \mapsto w^k\left(\frac{t}{\alpha_k}\right) \in \mathbb{R}$$

is a nontrivial periodic solution of Eq. (3.1) with $\alpha = \alpha_k$ and $g(\xi) = f'(0)\xi$ for all $\xi \in \mathbb{R}$. The minimal period is $T_{y^k} = \frac{2\pi\alpha_k}{\operatorname{Im}\zeta_k(\alpha_k)}$. From Corollary 3.2 we infer that the closed curves

$$Y^k : [0, T_{y^k}] \ni t \mapsto \begin{pmatrix} y^k(t) \\ \dot{y}^k(t) \end{pmatrix} \in \mathbb{R}^2 \quad \text{and} \quad X : [0, T_x] \ni t \mapsto \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathbb{R}^2$$

satisfy $0 \in \operatorname{int}(Y^k) \cap \operatorname{int}(X)$. Then we find $\beta > 0$ such that

$$\beta|Y^k| \subset |X| \cup \operatorname{ext}(X),$$

$$\beta|Y^k| \cap |X| \neq \emptyset,$$

$$r\beta|Y^k| \subset \operatorname{ext}(X) \qquad \text{for all } r > 1.$$

Consider $z : \mathbb{R} \ni t \mapsto \beta y^k(t) \in \mathbb{R}$ and $w : \mathbb{R} \ni t \mapsto z(\alpha_k t) \in \mathbb{R}$. Clearly, $V(w_t) = 2k$ for all $t \in \mathbb{R}$, and z is a T_{y^k}-periodic solution of Eq. (3.1) with $\alpha = \alpha_k$ and g as above. The closed curve

$$Z : [0, T_{y^k}] \ni t \mapsto \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} \in \mathbb{R}^2$$

satisfies $|Z| \subset |X| \cup \text{ext}(X)$, $|Z| \cap |X| \neq \emptyset$, and $r|Z| \subset \text{ext}(X)$ for all $r > 1$. Clearly, $g'(0) = f'(0)$. The fact $\lim_{\xi\to 0} \frac{\xi f'(\xi)}{f(\xi)} = 1$ and (H4) combined yield $\frac{\xi f'(\xi)}{f(\xi)} < 1$ for all $\xi > 0$, or equivalently, $\left(\frac{f(\xi)}{\xi}\right)' < 0$ for all $\xi > 0$. Using this and the equation $\lim_{\xi\to 0} \frac{f(\xi)}{\xi} = f'(0)$ we obtain $g(\xi) = f'(0)\xi > f(\xi)$ for all $\xi > 0$. Moreover,

$$\frac{g'(\xi)}{g(\xi)} = \frac{1}{\xi} > \frac{f'(\xi)}{f(\xi)} \qquad \text{for all } \xi > 0.$$

Applying Proposition 3.4 with $\alpha = \alpha_k$ and $g(\xi) = f'(0)\xi$ we get a contradiction. □

Corollary 3.6. *If hypotheses (H1), (H2), (H3) and (H4) hold then* $\alpha_2 > 1$, $\operatorname{Re}\lambda_2 < 0$, *and the periodic orbit* $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$ *of Theorem 2.3 is the only periodic orbit of Eq. (1.1).*

Proof. 1. The numbers θ from (H3) and $\operatorname{Im}\zeta_2(\alpha_2) = 4\pi - \arccos\frac{\mu}{f'(0)}$ both belong to the interval $(4\pi - \frac{\pi}{2}, 4\pi)$. Using the inequality $f'(0) < \frac{\mu}{\cos\theta}$ in (H3) we get

$$\alpha_2 > \frac{\operatorname{Im}\zeta_2(\alpha_2)}{\mu\sqrt{\frac{1}{(\cos\theta)^2} - 1}} = \frac{\operatorname{Im}\zeta_2(\alpha_2)}{-\mu\tan\theta} = \frac{\operatorname{Im}\zeta_2(\alpha_2)}{\theta}.$$

Also,

$$\cos(\operatorname{Im}\zeta_2(\alpha_2)) = \cos(4\pi - \operatorname{Im}\zeta_2(\alpha_2)) = \frac{\mu}{f'(0)} > \cos\theta,$$

and it follows that $\alpha_2 > 1$ which gives $\operatorname{Re}\lambda_2 < 0$, according to the remarks preceding Theorem 3.5.

2. Recall $\alpha_j < \alpha_{j+1}$ for $0 \neq j \in \mathbb{N}$ and apply Theorem 3.5. □

4. The global attractor is $\overline{W}$

The main result of this paper is the following theorem.

Theorem 4.1. *Under hypotheses (H1)–(H4) the set $\overline{W}$ is the global attractor of the semiflow F generated by Eq. (1.1).*

The remaining part of the paper contains the proof of this result. The proof consists of several parts which are formulated as propositions. Throughout this section we assume (H1)–(H4) hold. We emphasize that in the proof of Theorem 4.1 below hypothesis (H4), in particular the oddness of f, is not used explicitly; it enters only as one of the conditions which permit to apply Corollary 3.6.

In view of the results of Section 2 it remains to show

$$A \subseteq \overline{W}.$$

Note first that we have

$$\operatorname{Re}\lambda_2 < 0,$$

due to Corollary 3.6. It follows that the stationary point 0 is hyperbolic, and that W_{loc} is a full local unstable manifold of F at 0.

Next, recall the characterization $A = B$ from Proposition 2.2.

We need the following corollary of a general Poincaré–Bendixson theorem for monotone cyclic feedback systems due to Mallet-Paret and Sell [MPS2].

Proposition 4.2. *Let $x : \mathbb{R} \to \mathbb{R}$ be a bounded solution of Eq. (1.1). Then $\alpha(x)$ is either the orbit of a nonconstant periodic solution of Eq. (1.1), or for every solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 \in \alpha(x)$ the sets $\alpha(y)$ and $\omega(y_0)$ consist of stationary points of F.*

The analogue of Proposition 4.2 for ω-limit sets holds as well.

Proposition 4.3. *Let $x : \mathbb{R} \to \mathbb{R}$ be a bounded solution of Eq. (1.1). Then $\alpha(x) = \mathcal{O}$, or $\alpha(x) = \{0\}$, or $x(t) = \xi^-$ for all $t \in \mathbb{R}$, or $x(t) = \xi^+$ for all $t \in \mathbb{R}$.*

Proof. Suppose $\alpha(x) \neq \mathcal{O}$. We apply Proposition 4.2 and Corollary 3.6, and the fact that $\xi_-, 0, \xi_+$ are the only stationary points of F, and infer

$$\alpha(x) \cap \{0, \xi_-, \xi_+\} \neq \emptyset.$$

In case $\xi_- \in \alpha(x)$ the fact that ξ_- is stable implies that for every $\epsilon > 0$ and for every $t \in \mathbb{R}$ there exists $s \leq t$ with $||x_u - \xi_-|| < \epsilon$ for all $u \geq s$. It follows that $||x_t - \xi_-|| < \epsilon$, and then

$$x(t) = \xi^- \qquad \text{for all } t \in \mathbb{R}.$$

Analogously, we get in case $\xi_+ \in \alpha(x)$ that

$$x(t) = \xi^+ \qquad \text{for all } t \in \mathbb{R}.$$

In the remaining case

$$0 \in \alpha(x) \quad \text{and} \quad \alpha(x) \cap \{\xi_-, \xi_+\} = \emptyset.$$

Suppose there exists $\phi \in \alpha(x) \setminus \{0\}$. Consider the solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 = \phi$. By Proposition 4.2, we have $\alpha(y) \cup \omega(y_0) \subseteq \{0, \xi_-, \xi_+\}$. From $\alpha(y) \cup \omega(y_0) \subseteq \alpha(x)$ it follows that $\alpha(y) = \omega(y_0) = \{0\}$. Using that 0 is hyperbolic and W_{loc} is a local unstable manifold of F at 0, we find $s \in \mathbb{R}$ so that $y_u \in W_{loc}$ for all $u \leq s$. The definition of W gives $y_t \in W$ for all $t \in \mathbb{R}$. Theorem 2.3 and $\phi \in W \setminus \{0\}$ together imply $\omega(y_0) = \mathcal{O}$, or $\omega(y_0) = \{\xi_-\}$, or $\omega(y_0) = \{\xi_+\}$. All these possibilities contradict $\omega(y_0) = \{0\}$. □

In case $\phi \in A$ is the segment x_0 of a bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $\alpha(x) = \{0\}$ we obtain $\phi \in W = F([0, \infty) \times W_{loc})$ from $\lim_{t \to -\infty} x_t = 0$ and from the fact that W_{loc} is a local unstable manifold of F at the hyperbolic stationary point 0. If $\phi = \xi_-$ or $\phi = \xi_+$ then $\phi \in \overline{W}$ according to Theorem 2.3. Thus in view of the preceding proposition it remains to show that for every segment $\phi = x_t$, $t \in \mathbb{R}$, of a bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with

$$\alpha(x) = \mathcal{O}$$

we have

$$\phi \in \overline{W}.$$

In order to accomplish this we need several preliminary results. The first one is a version of Proposition 4.3 for ω-limit sets. Note that the domains of attraction to ξ_- and to ξ_+, namely the sets

$$A_- = \{\phi \in C : F(t, \phi) \to \xi_- \text{ as } t \to \infty\} \quad \text{and}$$
$$A_+ = \{\phi \in C : F(t, \phi) \to \xi_+ \text{ as } t \to \infty\},$$

are nonempty and open.

Proposition 4.4. *For $\phi \in C \setminus (A_- \cup A_+)$, either $\omega(\phi) = \mathcal{O}$ or $\omega(\phi) = \{0\}$.*

Proof. 1. If $\omega(\psi) \cap (A_- \cup A_+) \neq \emptyset$ for some $\psi \in C$, then $\omega(\psi) = \{\xi_-\}$ or $\omega(\psi) = \{\xi_+\}$. This follows from the definitions of the domains of attraction and of ω-limit sets.

2. Suppose $\phi \in C \setminus (A_- \cup A_+)$ and $\omega(\phi) \neq \mathcal{O}$.

2.1. Then $\omega(\phi) \cap \{\xi_-, \xi_+\} = \emptyset$ because otherwise, part 1 yields $\omega(\phi) = \{\xi_-\}$ or $\omega(\phi) = \{\xi_+\}$ which implies $\phi \in A_- \cup A_+$, a contradiction.

2.2. Proof of $0 \in \omega(\phi)$: Let $\chi \in \omega(\phi)$ be given. The version of Proposition 4.2 for ω-limit sets and Corollary 3.6 combined yield $\omega(\chi) \subseteq \{0, \xi_-, \xi_+\}$. Using $\omega(\chi) \subseteq \omega(\phi)$ and step 2.1 we infer $\omega(\chi) \subseteq \{0\}$, which gives $0 \in \omega(\chi) \subseteq \omega(\phi)$.

2.3. Proof of $\omega(\phi) \subseteq \{0\}$: Let $\chi \in \omega(\phi)$ and consider the solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 = \chi$. By the version of Proposition 4.2 for ω-limit sets and Corollary 3.6, $\alpha(y) \cup \omega(\chi) \subseteq \{0, \xi_-, \xi_+\}$. ¿From $\omega(\phi) \cap \{\xi_-, \xi_+\} = \emptyset$ and $\alpha(y) \cup \omega(\chi) \subseteq \omega(\phi)$ we get $\alpha(y) = \{0\} = \omega(\chi)$. The equation $\alpha(y) = \{0\}$ yields $\chi \in W = F(\mathbb{R}_+ \times W_{loc})$, compare arguments given before Proposition 4.4. Using Theorem 2.3 we obtain $\chi = 0$ or $\omega(\chi) = \mathcal{O}$ or $\omega(\chi) = \{\xi_-\}$ or $\omega(\chi) = \{\xi_+\}$. The last three possibilities contradict $\omega(\chi) = \{0\}$. Therefore, $\chi = 0$, and thus $\omega(\phi) = \{0\}$. $\square$

Next we want to exclude bounded solutions $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $\alpha(x) = \mathcal{O}$ and $\omega(x_0) = \{0\}$.

Recall that Q denotes the realified generalized eigenspace of the generator of the semigroup $(D_2F(t,0))_{t\geq 0}$ which is associated with the complement of the three leading eigenvalues $\lambda_0, \lambda_1, \overline{\lambda_1}$ in the spectrum.

Proposition 4.5. $V(Q \setminus \{0\}) \subseteq \{4, 6, \ldots, \infty\}$.

Proof. Suppose $0 \neq \phi \in Q$ and $V(\phi) \leq 2$. Proposition 2.4(iii) implies that for the solution $y : [-1, \infty) \to \mathbb{R}$ of the linear equation

$$\dot{y}(t) = -\mu y(t) + f'(0) y(t-1) \tag{2.2}$$

with $y_0 = \phi$ there exists $t \geq 1$ with $y_s \in R$ for all $s \geq t$. Using Proposition 2.4(ii), continuous dependence on initial data, and Eq. (2.2), we find a neighbourhood U of ϕ in C so that for all $\psi \in U$, $||D_2F(t,0)\psi - y_t||_{C^1}$ is so small that $V(D_2F(t,0)\psi) = V(y_t) \leq 2$, and consequently

$$V(D_2F(s,0)\psi) \leq 2 \qquad \text{for all } s \geq t. \tag{4.1}$$

Choose $\psi \in U \cap Q$ so that its projection χ onto the realified generalized eigenspace of $\{\lambda_2, \overline{\lambda_2}\}$ along the complementary realified generalized eigenspace is nonzero. Then $\chi \in Q$, and there exist reals $a \neq 0$ and b so that the solution $z : [-1, \infty) \to \mathbb{R}$ of Eq. (2.2) with $z_0 = \chi$ is given by

$$z(s) = a e^{(\operatorname{Re} \lambda_2) s} \cos((\operatorname{Im} \lambda_2) s + b) \qquad \text{for } s \geq -1.$$

Let $v : [-1, \infty) \to \mathbb{R}$ denote the solution of Eq. (2.2) with $v_0 = \psi$. The estimate on the complementary subspace (see e.g. [HVL] or [DGVLW]) implies that there exist $\epsilon > 0$ and $c \geq 0$ so that for all $s \geq 0$,

$$\begin{aligned} |e^{-(\operatorname{Re} \lambda_2) s} v(s) - a \cos((\operatorname{Im} \lambda_2) s + b)| &\leq ||e^{-(\operatorname{Re} \lambda_2) s} D_2F(s,0)(\psi - \chi)|| \\ &\leq c e^{-\epsilon s} ||\psi - \chi||. \end{aligned}$$

Using $3\pi < \operatorname{Im}\lambda_2 < 4\pi$ we infer that for some $s \geq t$ the segment v_s has more than 2 sign changes, in contradiction to (4.1). $\square$

Set $A_0 = \{\phi \in C : F(t, \phi) \to 0 \text{ as } t \to \infty\}$.

Proposition 4.6. $V(A_0 \setminus \{0\}) \subseteq \{4, 6, \ldots, \infty\}$.

Proof. Suppose $V(\phi) \leq 2$ for some $\phi \in A_0 \setminus \{0\}$. Set $x = x^\phi$. From Proposition 2.4 we obtain $t \geq 0$ so that $V(x_s) = V(x_t) \leq 2$ for all $s \geq t$. Proposition 2.5 for $b : [0, \infty) \ni u \mapsto \int_0^1 f'(sx(u-1))\, ds \in \mathbb{R}$ and the remarks following Proposition 2.4 show that there exist $t_1 \geq t$ and $k \geq 0$ so that for all $s \geq t_1$,

$$||x_{s-1}|| \leq k||x_s||.$$

Then, for all $s \geq t_1$ and for all $u \in [s-1, s]$,

$$\begin{aligned} |\dot{x}(u)| &\leq \mu|x(u)| + \max\{f'(\xi) : |\xi| \leq \max_{-1 \leq v} |x(v)|\}||x_{s-1}|| \\ &\leq \mu||x_s|| + \max\{f'(\xi) : |\xi| \leq \max_{-1 \leq v} |x(v)|\}k||x_s||, \end{aligned}$$

and it follows that the bounded sequence of the C^1-functions $||x_j||^{-1}x_j$, $1 \leq j \in \mathbb{N}$, has uniformly bounded derivatives. The Arzèla–Ascoli theorem yields a subsequence of points $\phi_k = ||x_{j_k}||^{-1}x_{j_k}$, $k \in \mathbb{N}$, which converges to some unit vector $\rho \in C$. Proposition 2.4(i) gives $V(\rho) \leq \liminf_{k\to\infty} V(\phi_k) \leq 2$. On the other hand, the points x_{j_k} with k sufficiently large belong to a local stable manifold W^s of the map $F(1, \cdot)$ at 0, which implies $\rho \in T_0 W^s = Q$. Then Proposition 4.5 gives $V(\rho) \geq 4$, a contradiction. $\square$

Corollary 4.7. *Solutions $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $\alpha(x) = \mathcal{O}$ and $\omega(x_0) = \{0\}$ do not exist.*

Proof. Suppose the contrary and consider a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $\alpha(x) = \mathcal{O}$ and $\omega(x_0) = \{0\}$. Proposition 4.6 implies $V(x_s) \geq 4$ for all $s \in \mathbb{R}$. On the other hand, there is a sequence $(s_j)_{-\infty}^0$ in $\mathbb{R}$ with $s_j \to -\infty$ and $x_{s_j} \to p_0$ as $j \to -\infty$. We have $p_1 \in R$ because of Proposition 2.4(iii) and periodicity. Using the continuity of $C \ni \phi \mapsto F(1, \phi) \in C^1$, we obtain $||x_{s_j+1} - p_1||_{C^1} \to 0$ as $j \to -\infty$. Then Proposition 2.4(ii) yields $V(x_{s_j+1}) = 2$ for all sufficiently large negative integers j, a contradiction. $\square$

In the next step we exclude solutions which are homoclinic with respect to the periodic orbit $\mathcal{O}$.

Proposition 4.8. *There is no solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) such that $x_0 \notin \mathcal{O}$ and $\alpha(x) = \omega(x_0) = \mathcal{O}$.*

Proof. 1. Suppose that there exists a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that $x_0 \notin \mathcal{O}$ and $\alpha(x) = \omega(x_0) = \mathcal{O}$. Then Proposition 2.2 gives $x_t \in A$ for all $t \in \mathbb{R}$.

First we show that

$$V(x_t - x_s) = 2 \qquad \text{for all } t, s \text{ in } \mathbb{R} \text{ with } t \neq s. \tag{4.2}$$

For all t, s in $\mathbb{R}$ with $t \neq s$ we have $x_t \in S$, $x_s \in S$, and $0 \neq x_t - x_s$. Consequently, $x_t - x_s \in C \setminus (K \cup (-K))$, and $V(x_t - x_s) \geq 2$. In order to prove $V(x_t - x_s) \leq 2$ we distinguish two cases.

2. The case $t \neq s$ and $x_{t-n\omega} - x_{s-n\omega} \not\to 0$ as $n \to \infty$: Using the compactness of $\mathcal{O}$ and $\alpha(x) = \mathcal{O}$, there exist a strictly increasing sequence $(n_k)_0^\infty$ and reals t', s' in $[0, \omega)$ so that $t' \neq s'$ and

$$x_{t-n_k\omega} \to p_{t'} \quad \text{and} \quad x_{s-n_k\omega} \to p_{s'} \qquad \text{as } k \to \infty.$$

As the topologies induced on A from C and C^1 are equivalent, convergence holds also in the C^1-norm. Recall $V(p_{t'} - p_{s'}) = 2$ and $p_{t'} - p_{s'} \in R$. Proposition 2.4(ii) yields $V(x_{t-n_k\omega} - x_{s-n_k\omega}) = 2$ for all sufficiently large $k \in \mathbb{N}$. Using Proposition 2.4(iii) and the remarks thereafter we infer $V(x_t - x_s) \leq 2$.

3. The case $t \neq s$ and $x_{t-n\omega} - x_{s-n\omega} \to 0$ as $n \to \infty$: There exist $\tau \in [0, \omega)$ and a strictly increasing sequence $(n_k)_0^\infty$ so that $x_{t-n_k\omega} \to p_\tau$ and $x_{s-n_k\omega} \to p_\tau$ as $k \to \infty$. For $\epsilon \in (0, \omega)$ we obtain $x_{t+\epsilon-n_k\omega} \to p_{\tau+\epsilon} \neq p_\tau$ for $k \to \infty$, and therefore $x_{t+\epsilon-n_k\omega} - x_{s-n_k\omega} \to p_{\tau+\epsilon} - p_\tau \neq 0$ as $k \to \infty$. For $0 < \epsilon < \min\{\omega, |t-s|\}$, part 2 is applicable for $t + \epsilon$ instead of t. We obtain $x_{t+\epsilon} - x_s \neq 0$ and $V(x_{t+\epsilon} - x_s) \leq 2$ for $0 < \epsilon < \min\{\omega, |t-s|\}$. Proposition 2.4(i) yields

$$V(x_t - x_s) \leq \liminf_{\epsilon \to 0^+} V(x_{t+\epsilon} - x_s) \leq 2.$$

4. $V(x_t) = 2$ for all $t \in \mathbb{R}$ holds since $p_0 \in R$, $V(p_0) = 2$, and there are sequences $(t_n)_0^\infty$ and $(s_n)_0^\infty$ so that $t_n \to \infty$, $s_n \to -\infty$, $||x_{t_n} - p_0||_{C^1} \to 0$ and $||x_{s_n} - p_0||_{C^1} \to 0$ as $n \to \infty$. This implies also $x_t \in R$ for all $t \in \mathbb{R}$. Then all zeros of x are simple and there is a sequence $(t^n)_{-\infty}^\infty$ such that, for every $n \in \mathbb{Z}$,

$$t^{n+1} - t^n < 1, \quad t^{n+2} - t^n > 1,$$

$$x(t^n) = 0, \quad \dot{x}(t^{2n}) > 0, \quad \dot{x}(t^{2n+1}) < 0,$$
$$x(t) > 0 \qquad \text{for all } t \in (t^{2n}, t^{2n+1}),$$
$$x(t) < 0 \qquad \text{for all } t \in (t^{2n-1}, t^{2n}).$$

Using Eq. (1.1) one finds $x(t^{2n} - 1) > 0$ and $x(t^{2n+1} - 1) < 0$ for all $n \in \mathbb{Z}$.

5. Claim: For every $n \in \mathbb{Z}$, $x(t^n - 1) \neq x(t^{n+1} - 1)$ and $x(t^n - 1) \neq x(t^{n+2} - 1)$. For every $n \in \mathbb{Z}$,

$$\text{we have} \quad x(t^{n+1} - 1) > x(t^{n+3} - 1) \quad \text{in case} \quad x(t^n - 1) < x(t^{n+2} - 1),$$

and

$$\text{we have} \quad x(t^{n+1}-1) < x(t^{n+3}-1) \quad \text{in case} \quad x(t^n-1) > x(t^{n+2}-1).$$

The sequence $(x(t^{2n}-1))_{-\infty}^{\infty}$ is strictly monotone.

Proof of the claim: Proposition 2.4 and (4.2) combined yield $x_t - x_s \in R$ for all t, s in $\mathbb{R}$ with $t \neq s$. Consequently, the continuous curve

$$X : \mathbb{R} \ni t \mapsto \begin{pmatrix} x(t) \\ x(t-1) \end{pmatrix} \in \mathbb{R}^2$$

is injective. This immediately implies the first assertion of the claim.

Suppose $x(t^n-1) < x(t^{n+2}-1)$ and $n = 2k$. Then $0 < x(t^{2k}-1) < x(t^{2k+2}-1)$, $x(t^{2k+1}-1) < 0$, $x(t^{2k+3}-1) < 0$, and $x(t) > 0$ for all $t \in (t^{2k}, t^{2k+1}) \cup (t^{2k+2}, t^{2k+3})$.

The restriction $X\big|_{[t^{2k},t^{2k+1}]}$ and the line segment $\lambda : [0,1] \ni s \mapsto sX(t^{2k}) + (1-s)X(t^{2k+1}) \in \mathbb{R}^2$ form a simple closed curve γ. The set

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 : u < 0, \text{ or } u = 0 \text{ and } v < x(t^{2k+1}-1), \text{ or } u = 0 \text{ and } v > x(t^{2k}-1) \right\}$$

belongs to $\mathrm{ext}(\gamma)$ since each of its points can be connected by parallels to the abscissa in $\mathbb{R}^2 \setminus |\gamma|$ to points with arbitrarily large negative first components. In particular, $X(t^{2k+2}) \in \mathrm{ext}(\gamma)$. The injectivity of X and the inequality $x(t) > 0$ for all $t \in (t^{2k+2}, t^{2k+3})$ combined give

$$X([t^{2k+2}, t^{2k+3})) \subset \mathrm{ext}(\gamma).$$

Suppose $x(t^{2k+1}-1) < x(t^{2k+3}-1)$. Then $X(t^{2k+3}) \in \lambda((0,1))$. The inequality $\dot{x}(t^{2k+3}) < 0$ implies that there exists $\epsilon \in (0, t^{2k+3}-t^{2k+2})$ so that $X(t^{2k+3}-\epsilon) \in \mathrm{int}(\gamma)$, a contradiction to $X([t^{2k+2}, t^{2k+3})) \subset \mathrm{ext}(\gamma)$. Therefore, $x(t^{2k+1}-1) > x(t^{2k+3}-1)$. The proofs for the other cases of the second statement in the claim are analogous.

The assertion about monotonicity follows from the second statement.

6. The properties of p imply that $\tau = 0$ is the unique zero of p in $[0, \omega)$ with $\dot{p}(\tau) > 0$, or equivalently, $p(\tau - 1) > 0$. From $\alpha(x) = \omega(x_0) = \mathcal{O} = \{p_t : t \in [0, \omega]\}$ and from the properties of the sequence $(t^{2n})_{-\infty}^{\infty}$ listed in part 4 it follows that

$$x_{t^{2n}} \to p_0 \qquad \text{as } |n| \to \infty.$$

In particular, $x(t^{2n}-1) \to p(-1)$ as $|n| \to \infty$, which contradicts the last statement in the claim of part 5. $\square$

In addition to Proposition 4.4, Corollary 4.7 and Proposition 4.8, the final step in the proof of Theorem 4.1 requires a result from [KWW] about the Floquet multipliers of the periodic solution p, i.e., about the spectrum σ_p of the compact operator $D_2F(\omega,p_0)$. We briefly recall the relevant facts: Every $\lambda\in\sigma_p\setminus\{0\}$ is an isolated point of σ_p, and an eigenvalue with finite-dimensional generalized eigenspace. We have $1\in\sigma_p$ and

$$D_2F(\omega,p_0)\dot{p}_0=\dot{p}_0;$$

the periodic solution is called hyperbolic if $\sigma_p\cap S^1_{\mathbb{C}}=\{1\}$ and if 1 is a simple eigenvalue (i.e., the generalized eigenspace of 1 is one-dimensional). For $\lambda\in\sigma_p\setminus\{0\}$ with $\operatorname{Im}\lambda\geq 0$ and for $r>0$ let $G_{\mathbb{R}}(\lambda)$, $C_{\leq r}$ and $C_{r<}$ denote the realified generalized eigenspaces of the spectral sets $\{\lambda,\overline{\lambda}\}$, $\{\zeta\in\sigma_p:|\zeta|\leq r\}$, and $\{\zeta\in\sigma_p:r<|\zeta|\}$, respectively. Then

$$C_{r<}=\bigoplus_{\lambda\in\sigma_p,\operatorname{Im}\lambda\geq 0,r<|\lambda|}G_{\mathbb{R}}(\lambda),$$

and we have the following result.

Proposition 4.9. *There exist $r_p\in(0,1)$, $\lambda_c\in(r_p,1]$, and $\lambda_u\in(1,\infty)$ so that*

$$\begin{aligned}
&\textit{either } \lambda_c<1,\ \{\lambda\in\sigma_p:r_p<|\lambda|\}=\{\lambda_c,1,\lambda_u\},\ \textit{and}\\
&\quad 1=\dim G_{\mathbb{R}}(\lambda_c)=\dim G_{\mathbb{R}}(1)=\dim G_{\mathbb{R}}(\lambda_u),\\
&\textit{or } \lambda_c=1,\ \{\lambda\in\sigma_p:r_p<|\lambda|\}=\{1,\lambda_u\},\ \textit{and}\\
&\quad 2=\dim G_{\mathbb{R}}(1),\ 1=\dim G_{\mathbb{R}}(\lambda_u).
\end{aligned}$$

In both cases,

$$C_{\leq r_p}\cap V^{-1}(\{0,2\})=\emptyset\quad\textit{and}\quad C_{r_p<}\setminus\{0\}\subset V^{-1}(\{0,2\}).$$

The remarks preceding Proposition 4.4 show that the following result completes the proof of Theorem 4.1.

Proposition 4.10. *For every bounded solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) with $\alpha(x)=\mathcal{O}$ and for every $t\in\mathbb{R}$,*

$$x_t\in\overline{W}.$$

Proof. 1. The case $\mathcal{O}$ is hyperbolic: The set

$$Y=C_{<1}\oplus G_{\mathbb{R}}(\lambda_u)$$

is a closed subspace of codimension 1 in C. The C^1-curve $t\mapsto p_t$ intersects p_0+Y transversally at $t=\omega$ since

$$D_1F(\omega,p_0)1=\dot{p}_\omega=\dot{p}_0\in G_{\mathbb{R}}(1).$$

As in Appendix I of [KWW] we obtain an open neighbourhood U of p_0 in C and a C^1-map

$$\tau : U \to \mathbb{R}$$

with $\tau(p_0) = \omega$ and $F(\tau(\phi), \phi) \in p_0 + Y$ for all $\phi \in U$, and p_0 is a hyperbolic fixed point of the Poincaré map

$$P : U \cap (p_0 + Y) \ni \phi \mapsto F(\tau(\phi), \phi) \in p_0 + Y.$$

In Section 11 of [KWW] the unstable set

$$W^u(\mathcal{O}) = F(\mathbb{R}^+ \times W^u(p_0, F(\omega, \cdot), N^u))$$

of the periodic orbit $\mathcal{O}$ was defined by means of a local unstable manifold $W^u(p_0, F(\omega, \cdot), N^u)$ of the period map $F(\omega, \cdot)$ at its fixed point p_0, and in Section 12 of [KWW] the equation

$$W^u(\mathcal{O}) = \operatorname{bd} W \setminus \{\xi_-, \xi_+\}$$

was established. Proposition V.1 of [KWW] guarantees that a local unstable manifold W^u of P at p_0 is contained in $F(\mathbb{R}^+ \times W^u(p_0, F(\omega, \cdot), N^u))$. It follows that

$$W^u \subset \overline{W},$$

and by invariance,

$$F(\mathbb{R}^+ \times W^u) \subset \overline{W}.$$

Now let a bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $\alpha(x) = \mathcal{O}$ be given. In the Appendix we show how to find a strictly increasing sequence $(t_j)_{-\infty}^{0}$ in $\mathbb{R}$ so that the points x_{t_j}, $j \in -\mathbb{N}$, form a trajectory of P with

$$x_{t_j} \to p_0 \qquad \text{as } j \to -\infty.$$

Hyperbolicity implies that for some $j_0 \in -\mathbb{N}$ all x_{t_j}, $j \leq j_0$, belong to W^u. Consequently, for every $t \in \mathbb{R}$,

$$x_t \in F(\mathbb{R}^+ \times W^u) \subset \overline{W}.$$

2. The case $\mathcal{O}$ is not hyperbolic: As the details are somewhat involved we first describe the idea of the proof. As in part 1 we consider a Poincaré map P associated with the periodic orbit $\mathcal{O}$ and with the fixed point p_0. In the present case there is a 2-dimensional local center-unstable manifold W^{cu} of P at p_0, which contains a one-dimensional local unstable manifold $W^u \subset \overline{W}$. For a given bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with

$\alpha(x) = \mathcal{O}$ we want to show that all segments x_t belong to $\overline{W}$. As in part 1 the equation $\alpha(x) = \mathcal{O}$ yields a strictly increasing sequence of reals t_j, $j \in -\mathbb{N}$, so that the segments x_{t_j} form a trajectory of P and converge to p_0 as $j \to -\infty$, but now we only obtain

$$x_{t_j} \in W^{cu}$$

for all integers $j \leq j_0$ with some $j_0 \in -\mathbb{N}$. Suppose the points x_{t_j} do not belong to W^u. Then we get unstable motion away from p_0 and W^u on one (open) side W^{cu}_+ of W^{cu}; for every ϕ in this side, there is a backward trajectory $(\phi_j)^0_{-\infty}$ of P with $\phi_0 = \phi$ so that

$$\phi_j \to p_0 \qquad \text{as } j \to -\infty. \tag{4.3}$$

There are points on the branches of $W^u \subset W^{cu}$ below and above a center manifold $W^c \subset W^{cu}$ of P at p_0 which belong to the open domains of attraction A_- and A_+ towards ξ_- and ξ_+, respectively. A continuity argument yields a point $\phi \in W^{cu}_+$ so that $F(t, \phi) \not\to \xi_-$, $F(t, \phi) \not\to \xi_+$ as $t \to \infty$. Then

$$\omega(\phi) = \mathcal{O} \quad \text{or} \quad \omega(\phi) = \{0\},$$

and (4.3) implies that there is a solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 = \phi$ and $\alpha(y) = \mathcal{O}$. As there are no nontrivial connections from $\mathcal{O}$ to $\mathcal{O}$ and no connections from $\mathcal{O}$ to 0, we obtain a contradiction, and have shown

$$x_{t_j} \in W^u \subset W^u(\mathcal{O}) \subset \overline{W}$$

for some $j \in -\mathbb{N}$. Using the flow F_W we get $x_t \in \overline{W}$ for all $t \in \mathbb{R}$.

We begin the description of the details.

2.1. Choose a basis vector ψ_u of $G_{\mathbb{R}}(\lambda_u)$ and a vector ψ_c so that

$$G_{\mathbb{R}}(1) = \mathbb{R}\dot{p}_0 \oplus \mathbb{R}\psi_c.$$

The set $Y = C_{<1} \oplus \mathbb{R}\psi_c \oplus \mathbb{R}\psi_u$ is a linear subspace of codimension 1 in C, and $\dot{p}_0 \notin Y$. Then consider an open neighbourhood U of p_0 and a Poincaré map

$$P : U \cap (p_0 + Y) \to p_0 + Y.$$

Choosing the neighbourhood U of p_0 small enough one can achieve that

$$U \cap (p_0 + Y) \cap \mathcal{O} = \{p_0\} \quad \text{and} \quad U \cap \{0, \xi_-, \xi_+\} = \emptyset.$$

There are a local unstable manifold W^u of P at p_0, $\lambda \in (\frac{1}{\lambda_u}, 1)$ and an open neighbourhood N_Y^u of 0 in Y so that for every backward trajectory $(\phi_j)_{-\infty}^0$ of P with

$$\lambda^j(\phi_j - p_0) \in N_Y^u \qquad \text{for all } j \in -\mathbb{N}$$

and

$$\lambda^j(\phi_j - p_0) \to 0 \qquad \text{as } j \to -\infty$$

we have

$$\phi_0 \in W^u.$$

As in part 1

$$W^u \subset \overline{W}.$$

2.2. Claim: Every neighbourhood of p_0 in W^u contains points of A_- and points of A_+.

Proof: Proposition 12.5 of [KWW] guarantees that the local unstable manifold of a certain Poincaré map P_H associated with $\mathcal{O}$ contains points from A_- as well as from A_+. An application of Theorem V.3 in [KWW] to P_H then shows that $W^u(\mathcal{O}) \cap A_- \neq \emptyset \neq W^u(\mathcal{O}) \cap A_+$. By Theorem V.3 in [KWW] once again, now applied to P, we get $W^u(\mathcal{O}) = F(\mathbb{R}_+ \times W^u)$, and it follows that

$$W^u \cap A_- \neq \emptyset \neq W^u \cap A_+.$$

Finally, use that each point in $W^u \cap A_-$ and $W^u \cap A_+$ extends to a backward trajectory $(\phi_j)_{-\infty}^0$ of P which converges to p_0 as $j \to -\infty$; it follows that every neighbourhood of p_0 contains points from A_- and A_+.

2.3. Consider a C^1-smooth center-unstable manifold W^{cu} of the Poincaré map P at p_0 as in [KWW]. It is not difficult to see that there are open neighbourhoods N and V of p_0 in C with the following properties:

$P(N \cap W^{cu}) = V \cap W^{cu}$, and P defines a C^1-diffeomorphism from $N \cap W^{cu}$ onto $V \cap W^{cu}$. Let P_{cu}^{-1} denote the inverse.

Every trajectory $(\phi_n)_{-\infty}^0$ of P in N has all its points in W^{cu}.

The projection $\mathrm{Pr}_{Y_{1\le}} : Y \to Y$ along $Y_{<1} = C_{<1}$ onto $Y_{1\le} = \mathbb{R}\psi_c \oplus \mathbb{R}\psi_u$ defines a C^1-diffeomorphism from $(V \cap W^{cu}) - p_0$ onto an open neighbourhood $N_{1\le}$ of 0 in $Y_{1\le}$.

The map

$$T_1 : V \cap W^{cu} \ni \phi \mapsto \mathrm{Pr}_{Y_{1\le}}(\phi - p_0) \in Y_{1\le}$$

satisfies $DT_1(p_0) = \mathrm{id}_{Y_{1\le}}$. Choose an open neighbourhood V_1 of p_0 in V so small that

$$P_{cu}^{-1}(V_1 \cap W^{cu}) \subset V \cap W^{cu},$$

and set

$$U_1 = T_1(V_1 \cap W^{cu})(\subset N_{1\leq}).$$

Let $T_1^{-1} : N_{1\leq} \to W^{cu}$ denote the inverse of T_1. The transformed map

$$P_1^{-1} : U_1 \ni \phi \mapsto T_1(P_{cu}^{-1}(T_1^{-1}(\phi))) \in Y_{1\leq}$$

is C^1-smooth with $P_1^{-1}(U_1) \subset N_{1\leq}$, $P_1^{-1}(0) = 0$. The eigenvalues of $DP_1^{-1}(0)$ are 1 and $\lambda_u^{-1}(< 1)$, the corresponding realified eigenspaces are $Y_1 = \mathbb{R}\psi_c$ and $Y_{1<} = \mathbb{R}\psi_u$, respectively. Using a local center manifold of P_1^{-1} at 0 and a local stable manifold of P_1^{-1} at 0 we find a C^1-diffeomorphism T_2 from an open neighbourhood U_2 of 0 in U_1 onto an open neighbourhood $\tilde{U}_{\mathcal{R}}$ of 0 in $Y_{1\leq}$, with $DT_2(0) = \mathrm{id}_{Y_{1\leq}}$, and an open neighbourhood $U_{\mathcal{R}}$ of 0 in $\tilde{U}_{\mathcal{R}}$ so that T_2 and the restricted inverse $T_2^{-1} : U_{\mathcal{R}} \to Y_{1\leq}$ satisfy

$$P_1^{-1}(T_2^{-1}(U_{\mathcal{R}})) \subset U_2,$$

and so that the transformed map

$$P_2^{-1} : U_{\mathcal{R}} \ni \phi \mapsto T_2(P_1^{-1}(T_2^{-1}\phi)) \in Y_{1\leq}$$

has the following properties:

$$P_2^{-1}(0) = 0, \quad DP_2^{-1}(0)\psi_c = \psi_c, \quad DP_2^{-1}(0)\psi_u = \lambda_u^{-1}\psi_u,$$

$$P_2^{-1}(U_{\mathcal{R}} \cap \mathbb{R}\psi_u) \subset \mathbb{R}\psi_u, \quad P_2^{-1}(U_{\mathcal{R}} \cap \mathbb{R}\psi_c) \subset \mathbb{R}\psi_c.$$

So we can apply Appendix IV of [KWW] to the map

$$f_p = P_2^{-1}$$

and to the 1-dimensional spaces $Z = \mathbb{R}\psi_u$ and $\mathbb{R}\psi_c$. Fix $\alpha \in (\frac{1}{\lambda_u}, \lambda)$. Choose a new (equivalent) norm $||\cdot||_s$ on $Y_{1\leq}$ so that the projections $Y_{1\leq} \to Y_{1\leq}$ along Z onto $\mathbb{R}\psi_c$ and along $\mathbb{R}\psi_c$ onto Z have norm 1 (with respect to $||\cdot||_s$). Clearly, $||Df_p(0)\phi||_s = \frac{1}{\lambda_u}||\phi||_s$ for all $\phi \in Z$. Choose $\epsilon \in (0,1)$ so that the quadrangle

$$\mathcal{R} = (-\epsilon, \epsilon)\psi_u + (-\epsilon, \epsilon)\psi_c$$

is contained in $U_{\mathcal{R}}$, and

$$||f_p(\phi)||_s \leq \alpha||\phi||_s \qquad \text{for all } \phi \in (-\epsilon, \epsilon)\psi_u.$$

Set

$$N_Z = (-\epsilon, \epsilon)\psi_u \quad \text{and} \quad N_c = (-\epsilon, \epsilon)\psi_c.$$

Incidentally, let us establish the following relation between Z and W^u.

2.4. Claim: The C^1-map

$$T^{-1} : U_{\mathcal{R}} \ni \phi \mapsto T_1^{-1}(T_2^{-1}(\phi)) \in p_0 + Y$$

defines a C^1-diffeomorphism from an open subset $(-\epsilon_u, \epsilon_u)\psi_u$, $\epsilon_u \in (0, \epsilon)$, onto an open neighbourhood of p_0 in W^u.

Proof: 2.4.1. $DT^{-1}(0)\psi_u \neq 0$ since T_1 and T_2 are C^1-diffeomorphisms. $T^{-1}(0) = p_0$.

2.4.2. f_p maps N_Z into itself, with

$$||f_p(\phi) - f_p(0)||_s \leq \alpha||\phi - 0||_s \qquad \text{on } N_Z.$$

2.4.3. T_1 is given by a translation, followed by a linear continuous map. So it is Lipschitz continuous. T_2 and T^{-1} are C^1-maps, and therefore locally Lipschitz continuous. It follows that there exist $\epsilon_1 \in (0, \epsilon)$ and constants $c_1 > 0$, $c_2 > 0$ with

$$||T^{-1}(\phi) - T^{-1}(\psi)|| \leq c_1||\phi - \psi||_s \qquad \text{on } (-\epsilon_1, \epsilon_1)\psi_u,$$

and

$$||T_2(T_1(\phi)) - T_2(T_1(\psi))||_s \leq c_2||\phi - \psi|| \qquad \text{on } T^{-1}((-\epsilon_1, \epsilon_1)\psi_u).$$

2.4.4. Define $T : T^{-1}((-\epsilon_1, \epsilon_1)\psi_u) \to (-\epsilon_1, \epsilon_1)\psi_u$ by

$$T(\psi) = T_2(T_1(\psi)).$$

Choose $\rho >$ so that $||\psi|| < \rho$ and $\psi \in Y$ imply $\psi \in N_Y^u$. Then choose $\epsilon_2 \in (0, \epsilon_1)$ so that for all $\phi \in (-\epsilon_2, \epsilon_2)\psi_u$, the element $\psi = T^{-1}(\phi)$ satisfies

$$c_1 c_2||\psi - p_0|| < \rho.$$

2.4.5. Let $\phi \in (-\epsilon_2, \epsilon_2)\psi_u$. Set $\phi_j = f_p^j(\phi)$, for $j \in \mathbb{N}$, and $\psi_j = T^{-1}(\phi_{-j})$, for $j \in -\mathbb{N}$. Then $(\psi_j)_{-\infty}^0$ is a trajectory of P, and for all $j \in -\mathbb{N}$ we have

$$\begin{aligned}||\lambda^j(\psi_j - p_0)|| &= \lambda^j||T^{-1}(\phi_{-j}) - T^{-1}(0)|| = \lambda^j||T^{-1}(f_p^{-j}(\phi)) - T^{-1}(0)|| \\ &\leq \lambda^j c_1 \alpha^{-j}||\phi - 0||_s \leq c_1 \left(\frac{\alpha}{\lambda}\right)^{-j} c_2||\psi_0 - p_0||.\end{aligned}$$

Using $\frac{\alpha}{\lambda} < 1$ we infer

$$\lambda^j(\psi_j - p_0) \to 0 \qquad \text{as } j \to -\infty$$

and

$$||\lambda^j(\psi_j - p_0)|| < \rho, \qquad \text{or} \quad \lambda^j(\psi_j - p_0) \in N_Y^u.$$

It follows that

$$\psi_0 = T^{-1}(\phi) \quad \text{belongs to} \quad W^u.$$

2.4.6. We have $T^{-1}((-\epsilon_2, \epsilon_2)\psi_u) \subset W^u$ by part 2.4.5. Hence $DT^{-1}(0)T_0(-\epsilon_2, \epsilon_2)\psi_u \subset T_{p_0}W^u$. Using the result of part 2.4.1 and $\dim T_{p_0}W^u = 1$ we conclude that T^{-1} defines a C^1-diffeomorphism from an open neighbourhood $(-\epsilon_u, \epsilon_u)\psi_u$, $0 < \epsilon_u < \epsilon_2$, onto an open neighbourhood of p_0 in W^u.

2.5. We define open sides of the quadrangle $\mathcal{R}$ by

$$\mathcal{R}_+ = N_Z + (0, \epsilon)\psi_c, \quad \mathcal{R}_- = N_Z + (-\epsilon, 0)\psi_c,$$

introduce slopes of vectors $\rho = \rho_Z + \rho_c$ with $\rho_Z \in Z$ and $0 \neq \rho_c \in \mathbb{R}\psi_c$ by

$$\mathrm{sl}(\rho) = \frac{||\rho_Z||_s}{||\rho_c||_s},$$

choose $h > 0$, and consider cones

$$\begin{aligned} K_h^+ &= \{\rho \in Y_{1\leq} : \mathrm{Pr}_{\mathbb{R}\psi_c}\, \rho \in (0, \infty)\psi_c,\ \mathrm{sl}(\rho) < h\}, \\ K_h^- &= \{\rho \in Y_{1\leq} : \mathrm{Pr}_{\mathbb{R}\psi_c}\, \rho \in (-\infty, 0)\psi_c,\ \mathrm{sl}(\rho) < h\} \end{aligned}$$

as in Appendix IV of [KWW].

We repeat Proposition IV.1 of [KWW]:

For every $h > 0$ there exists an integer $n_1(h) \geq 4$ such that for each

$$\chi \in \frac{1}{n_1(h)}(N_Z + N_c)$$

we have

$$f_p\left(\frac{1}{n_1(h)}(N_Z + N_c) \cap (\chi + K_h^+)\right) \subset f_p(\chi) + K_h^+$$

and

$$f_p\left(\frac{1}{n_1(h)}(N_Z + N_c) \cap (\chi + K_h^-)\right) \subset f_p(\chi) + K_h^-.$$

Claim: f_p maps $\frac{1}{n_1(h)}\mathcal{R}_+$ into $Z + (0, \infty)\psi_c$ and $\frac{1}{n_1(h)}\mathcal{R}_-$ into $Z + (-\infty, 0)\psi_c$.

Proof: Let $\phi \in \frac{1}{n_1(h)}\mathcal{R}_+$. Then $\phi = \phi_Z + a\psi_c$ with $\phi_Z \in Z \cap \frac{1}{n_1(h)}(N_Z + N_c)$ and $0 < a < \epsilon$. In particular, $\phi \in \phi_Z + K_h^+$. Using Proposition IV.1 in [KWW] and $f_p(Z \cap \mathcal{R}) \subset Z$

we infer $f_p(\phi) \in f_p(\phi_Z) + K_h^+ \subset Z + K_h^+ \subset Z + (0,\infty)\psi_c$. The proof for $\phi \in \frac{1}{n_1(h)}\mathcal{R}_-$ is analogous. (So, f_p does not interchange the open sides of $\frac{1}{n_1(h)}\mathcal{R}$.)

2.6. Let a bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) be given with

$$\alpha(x) = \mathcal{O}.$$

In the Appendix we show how to find a trajectory $(\phi_j)_{-\infty}^0$ of P in N with

$$\phi_j \to p_0 \qquad \text{as } j \to -\infty$$

which consists of segments of x; all ϕ_j, $j \in -\mathbb{N}$, belong to W^{cu}. We may assume that for every $j \in -\mathbb{N}$, $\phi_j \in V$, $T_1(\phi_j) \in U_2$, and $\psi_{-j} = T_2(T_1(\phi_j))$ belongs to $\frac{1}{n_1(h)}\mathcal{R}$. Then $(\psi_j)_0^\infty$ is a trajectory of f_p in $\mathcal{R}$ with

$$\psi_j \to 0 \qquad \text{as } j \to \infty.$$

Recall from the description at the beginning of part 2 that the main step of the proof is to show $\phi_j \in W^u$ for some $j \in -\mathbb{N}$. This follows from part 2.4 provided we find $j \in \mathbb{N}$ with $\psi_j \in (-\epsilon_u, \epsilon_u)\psi_u \subset Z$. In the next step we show $\psi_0 \in Z$.

2.7. Suppose

$$\psi_0 \in \mathcal{R} \setminus Z.$$

2.7.1. Claim: Either there exists a trajectory $(\chi_j)_0^\infty$ of f_p in $(0,\epsilon)\psi_c$ with $\chi_j \to 0$ as $j \to \infty$, or there exists a trajectory $(\chi_j)_0^\infty$ of f_p in $(-\epsilon, 0)\psi_c$ with $\chi_j \to 0$ as $j \to \infty$; in both cases $||\chi_{j+1}||_s < ||\chi_j||_s$ for all $j \in \mathbb{N}$.

Proof: The claim in part 2.5 yields $\psi_j \in \mathcal{R}\setminus Z$ for all $j \in \mathbb{N}$. Proposition IV.3 of [KWW] applies and yields $\delta \in (0,\epsilon)$ so that for all $r \in (0,\delta)$, $f_p(r\psi_c) = \hat{r}\psi_c$ with $0 < \hat{r} < r$, or for all $r \in (-\delta, 0)$, $f_p(r\psi_c) = \hat{r}\psi_c$ with $r < \hat{r} < 0$. Consider the first case. Set $r = \frac{\delta}{2}$. The trajectory $(f_p^j(r\psi_c))_0^\infty$ of f_p in $(0,\epsilon)\psi_c$ converges to a point $r_0\psi_c$ with $0 \le r_0 < r$. In case $0 < r_0$, $r_0\psi_c$ is a fixed point of f_p in $\mathcal{R} \setminus \{0\}$, and $\rho_0 = T_1^{-1}(T_2^{-1}(r_0\psi_c))$ is a fixed point of P in $U \cap (p_0 + Y) \setminus \{p_0\}$. By $U \cap \{0, \xi_-, \xi_+\} = \emptyset$, it follows that ρ_0 defines a nontrivial periodic solution of Eq. (1.1) whose orbit is $\mathcal{O}$, due to uniqueness; the relations $\rho_0 \in \mathcal{O} \cap U \cap (p_0 + Y) = \{p_0\}$ yield a contradiction. Therefore,

$$r_0 = 0.$$

Set $\chi_j = f_p^j(r\psi_c)$, for $j \in \mathbb{N}$. The estimates $||\chi_{j+1}||_s < ||\chi_j||_s$, $j \in \mathbb{N}$, are obvious. The proof for the second case is analogous.

2.7.2. Let $\mathcal{R}_o$, $o \in \{+,-\}$, denote the open side of $\mathcal{R}$ so that $\mathbb{R}\psi_c \cap \mathcal{R}_o$ contains a trajectory $(\chi_j)_0^\infty$ of f_p with $\chi_j \to 0$ as $j \to \infty$.

Claim: There exists an integer $n > 0$ so that for every $\psi \in \frac{1}{n}\mathcal{R}_o$ there is a trajectory $(\psi_j)_0^\infty$ of f_p in $\mathcal{R}$ with $\psi_0 = \psi$ and $\psi_j \to 0$ as $j \to \infty$.

Proof for $o = +$ (The proof for $o = -$ is analogous): There exists $k \in \mathbb{N}$ with

$$(\chi_j + K_h^-) \cap (Z + (0,\infty)\psi_c) \subset \frac{1}{n_1(h)}\mathcal{R}_+ \qquad \text{for all integers } j \geq k.$$

Consequently,

$$(\chi_j + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+ \subset \frac{1}{n_1(h)}\mathcal{R}_+ \qquad \text{for } k \leq j \in \mathbb{N}.$$

Using the result of Proposition IV.1 in [KWW] and the claim in part 2.5 we find

$$\begin{aligned} f_p\left((\chi_j + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+\right) &\subset (\chi_{j+1} + K_h^-) \cap (Z + (0,\infty)\psi_c) \\ &\subset (\chi_{j+1} + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+ \qquad \text{for } k \leq j \in \mathbb{N}. \end{aligned}$$

The sets $(\chi_{j+1} + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+$, $k \leq j \in \mathbb{N}$, are decreasing (use $||\chi_{j+1}||_s < ||\chi_j||_s$), have empty intersection, and converge to 0 as $j \to \infty$. It follows that for every $\psi \in (\chi_k + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+$ there is a trajectory $(\psi_l)_0^\infty$ of f_p in $\frac{1}{n_1(h)}\mathcal{R}_+ \subset \mathcal{R}$ with $\psi_0 = \psi$ and $\psi_l \to 0$ as $l \to \infty$. Choosing an integer $n > n_1(h)$ so that $\frac{1}{n}\mathcal{R}_+ \subset (\chi_k + K_h^-) \cap \frac{1}{n_1(h)}\mathcal{R}_+$, the proof of the claim is complete.

2.7.3. Consider the open neighbourhood $\frac{1}{n}(-\epsilon_u, \epsilon_u)\psi_u$ of 0 in $(-\epsilon_u, \epsilon_u)\psi_u$. The claims in parts 2.2 and 2.4 combined yield that there exist points

$$\phi_- \quad \text{and} \quad \phi_+ \quad \text{in} \quad \frac{1}{n}(-\epsilon_u, \epsilon_u)\psi_u \subset \frac{1}{n}(-\epsilon, \epsilon)\psi_u$$

so that $T^{-1}(\phi_-) \in A_-$, $T^{-1}(\phi_+) \in A_+$. Then the openness of A_- and of A_+ and the continuity of T^{-1} and the fact that ϕ_-, ϕ_+ belong to the boundary of $\frac{1}{n}\mathcal{R}_o$ in $Y_{1\leq}$ combined yield that there exist points ψ_-, ψ_+ in $\frac{1}{n}\mathcal{R}_o$ with

$$T^{-1}(\psi_-) \in A_-, \quad T^{-1}(\psi_+) \in A_+.$$

So, the open subsets $\{\psi \in \frac{1}{n}\mathcal{R}_o : T^{-1}(\psi) \in A_-\}$, $\{\psi \in \frac{1}{n}\mathcal{R}_o : T^{-1}(\psi) \in A_+\}$ of $\frac{1}{n}\mathcal{R}_o$ are nonempty. By connectedness, there exists $\psi \in \frac{1}{n}\mathcal{R}_o$ with

$$T^{-1}(\psi) \in C \setminus (A_- \cup A_+).$$

Set $\phi = T^{-1}(\psi)$. Then $\phi \neq p_0$ (since $\psi \in \frac{1}{n}\mathcal{R}_o$ and $0 \notin \frac{1}{n}\mathcal{R}_o$), and $\phi \in C \setminus \{0\}$.

Using the claim of part 2.7.2 we infer that ϕ extends to a trajectory $(\tilde{\phi}_j)_{-\infty}^0$ of P with $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}_j \to p_0$ as $j \to -\infty$. There is a solution $y : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $y_0 = \phi$. We deduce

$$\alpha(y) = \mathcal{O}.$$

Proposition 4.4 and $\phi \notin A_- \cup A_+$ combined give the alternative

$$\omega(\phi) = \mathcal{O} \quad \text{or} \quad \omega(\phi) = \{0\}.$$

Since $\phi \in U \cap (p_0 + Y) \setminus \{p_0\}$ and $U \cap (p_0 + Y) \cap \mathcal{O} = \{p_0\}$, we obtain $\mathcal{O} \cap \{y_t : t \in \mathbb{R}\} = \emptyset$. The absence of nontrivial solutions homoclinic to $\mathcal{O}$ (Proposition 4.8) rules out the possibility $\omega(\phi) = \mathcal{O}$. Corollary 4.7 excludes the remaining possibility. So, altogether we arrived at a contradiction; (4.4) is false, and we have

$$\psi_0 \in \mathcal{R} \cap Z = (-\epsilon, \epsilon)\psi_u.$$

2.8. As f_p contracts $(-\epsilon, \epsilon)\psi_u$ we obtain $j \in \mathbb{N}$ with $\psi_j \in (-\epsilon_u, \epsilon_u)\psi_u$. By the claim of part 2.4, $\phi_{-j} = T^{-1}(\psi_j) \in W^u \subset \overline{W}$. Using the invariance properties of $\overline{W}$ and the injectivity of all maps $F(t, \cdot)$, $t \geq 0$, we finally conclude that $x_t \in \overline{W}$ for all $t \in \mathbb{R}$. $\square$

Appendix: Periodic orbits as α-limit sets and backward trajectories of Poincaré maps

We provide the details of the proof of a result which is familiar for ordinary differential equations, but seems not available in quotable form for delay differential equations. In the sequel we consider a strictly monotone C^1-function $f : \mathbb{R} \to \mathbb{R}$, a real $\mu > 0$, a nonconstant periodic solution $q : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with minimal period $t_q > 1$, and a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that x is bounded on $(-\infty, 0]$ and has α-limit set

$$\mathcal{Q} = \{q_t : t \in \mathbb{R}\}.$$

Let $F : \mathbb{R}^+ \times C \to C$ be the semiflow of Eq. (1.1) as in Section 2. Each map $F(t, \cdot)$, $t \geq 0$, is injective. Let a closed subspace H of C with codimension 1 be given so that

$$\dot{q}_0 = \dot{q}_{t_q} = D_1F(t_q, q_0)1 \in C \setminus H.$$

We recall the construction of a Poincaré map associated with $\mathcal{Q}, t_q, H$, and $q_0 \in q_0 + H$. There is a continuous linear functional $\eta : C \to \mathbb{R}$ with

$$H = \eta^{-1}(0).$$

An application of the implicit function theorem to the equation

$$\eta(F(t, \phi) - q_0) = 0$$

close to its solution $(t_q, q_0) \in \mathbb{R} \times C$ yields an open neighbourhood N of q_0 in C, $\epsilon \in (0, t_q/2)$, and a C^1-map $\tau : N \to (t_q - \epsilon, t_q + \epsilon)$ so that $\tau(q_0) = t_q$ and $F(\tau(\phi), \phi) \in q_0 + H$ for all $\phi \in N$, and for every $(t, \phi) \in (t_q - \epsilon, t_q + \epsilon) \times N$ with $F(t, \phi) \in q_0 + H$, $t = \tau(\phi)$.

The Poincaré map associated with $\mathcal{Q}, t_q, H, q_0$ (and N, ϵ, τ) is the map

$$P : N \cap (q_0 + H) \ni \phi \mapsto F(\tau(\phi), \phi) \in q_0 + H.$$

Proposition A.1. *There is a strictly increasing sequence $(t_j)_{-\infty}^0$ with $t_j \to -\infty$ as $j \to -\infty$ so that the points x_{t_j}, $j \in -\mathbb{N}$, form a trajectory of P with $x_{t_j} \to q_0$ as $j \to -\infty$.*

Proof. 1. Claim: $\text{dist}(x_t, \mathcal{Q}) \to 0$ as $t \to -\infty$.

Proof: Otherwise there are $\epsilon_0 > 0$ and a sequence $(t_j)_{-\infty}^0$ with $t_j \to -\infty$ as $j \to -\infty$ and $\text{dist}(x_{t_j}, \mathcal{Q}) \geq \epsilon_0$ for all $j \in -\mathbb{N}$. The boundedness of x on $(-\infty, 0]$, Eq. (1.1), and the

Arzèla–Ascoli theorem combined yield a subsequence $(\phi_k)_{-\infty}^0$ of $(x_{t_j})_{-\infty}^0$ which converges to some $\phi \in \mathcal{Q}$ as $k \to -\infty$. On the other hand,

$$||\phi - \phi_k|| \geq \inf_{0 \leq t \leq t_q} ||q_t - \phi_k|| = \mathrm{dist}(\phi_k, \mathcal{Q}) \geq \epsilon_0$$

for all $k \in -\mathbb{N}$, which is a contradiction.

2. Claim: There exists an open neighbourhood $N_0 \subset N$ of q_0 in C with

$$\overline{N_0} \cap (q_0 + H) \cap \mathcal{Q} = \{q_0\}.$$

Proof: We have $q_t \notin q_0 + H$ for $0 < |t - t_q| < \epsilon$. By periodicity, $q_t \notin q_0 + H$ for all $t \in (0, \epsilon) \cup (t_q - \epsilon, t_q)$. There is an open neighbourhood $N_{01} \subset N$ of q_0 in C with $\emptyset = N_{01} \cap \{q_t : \epsilon \leq t \leq t_q - \epsilon\}$. It follows that

$$\mathcal{Q} \setminus \{q_0\} \subset C \setminus (N_{01} \cap (q_0 + H)).$$

Choose an open ball N_0 centered at q_0 with $\overline{N_0} \subset N_{01}$.

3. Claim: There exists an open neighbourhood $N_1 \subset N_0$ of q_0 in C and $\epsilon_1 \in (0, \epsilon)$ so that for every $t \leq 0$ with $x_t \in N_1 \cap (q_0 + H)$, and for all $s \in (-\epsilon_1, \epsilon_1) \setminus \{0\}$,

$$x_{t+s} \notin q_0 + H.$$

Proof: Otherwise there are sequences $(t_j)_0^\infty$ in $(-\infty, 0]$ and $(s_j)_0^\infty$ in $\mathbb{R} \setminus \{0\}$ with $x_{t_j} \in N_0 \cap (q_0 + H)$ and $x_{t_j + s_j} \in q_0 + H$ for all $j \in \mathbb{N}$, and $x_{t_j} \to q_0$, $s_j \to 0$ as $j \to \infty$. Using the mean value theorem we find a sequence $(\tilde{s}_j)_0^\infty$ in $\mathbb{R} \setminus \{0\}$ so that $\tilde{s}_j \to 0$ as $j \to \infty$ and

$$\eta(\dot{x}_{t_j + \tilde{s}_j}) = \frac{d}{ds}(s \mapsto \eta(x_s))(t_j + \tilde{s}_j) = 0 \qquad \text{for all } j \in \mathbb{N}.$$

The sequence of the functions

$$[-3, 0] \ni s \mapsto x(t_j + s) \in \mathbb{R}, \qquad j \in \mathbb{N},$$

and the sequence of their derivatives are bounded. The Arzèla–Ascoli theorem yields a subsequence given by a strictly increasing map $\chi : \mathbb{N} \to \mathbb{N}$, which converges uniformly on $[-3, 0]$ to a continuous function. In particular, there exists $\phi \in C$ with $x_{t_{\chi(j)} - 2} \to \phi$ for $j \to \infty$. It follows that $x_{t_{\chi(j)}} \to F(2, \phi)$ as $j \to \infty$. Recall $x_{t_j} \to q_0$. Consequently, $q_0 = F(2, \phi)$, and by injectivity of $F(2, \cdot)$,

$$\phi = q_{-2}.$$

Furthermore, we obtain from continuous dependence on initial data that the functions

$$[-3,1] \ni s \mapsto x(t_{\chi(j)}+s) \in \mathbb{R}, \qquad j \in \mathbb{N},$$

converge uniformly to $q|_{[-3,1]}$ as $j \to \infty$. Using Eq. (1.1) we get that the derivatives

$$[-2,1] \ni s \mapsto \dot{x}(t_{\chi(j)}+s) \in \mathbb{R}, \qquad j \in \mathbb{N},$$

converge uniformly to $\dot{q}|_{[-2,1]}$ as $j \to \infty$. In particular, the numbers

$$|\eta(\dot{q}_{\tilde{s}_{\chi(j)}})| = |\eta(\dot{q}_{\tilde{s}_{\chi(j)}} - \dot{x}_{t_{\chi(j)}+s_{\chi(j)}})| \le ||\eta|| \max_{-2\le s\le 1} |\dot{q}(s) - \dot{x}(t_{\chi(j)}+s)|$$

tend to zero as $j \to \infty$. It follows that $\eta(\dot{q}_0) = 0$, in contradiction to $\dot{q}_0 \notin H$.

4. Claim: There exists an open neighbourhood $N_2 \subset N_1$ of q_0 in C so that for all $t \le 0$ with $x_t \in N_2 \cap (q_0 + H)$ and for $0 < s < \tau(x_t)$ we have

$$F(s, x_t) \notin N_2 \cap (q_0 + H).$$

Proof: Find disjoint open neighbourhoods $N_{21} \subset N_1$ of q_0 and N_* of the closed set $\{q_t : \epsilon_1 \le t \le t_q - \epsilon\}$. Continuous dependence on initial data permits to find an open neighbourhood $N_2 \subset N_{21}$ of q_0 so that for every $\phi \in N_2 \cap (q_0 + H)$ and every $t \in [\epsilon_1, t_q - \epsilon]$,

$$F(t, \phi) \in N_*.$$

In particular,

$$F(t, \phi) \notin N_2 \qquad \text{on } [\epsilon_1, t_q - \epsilon] \times N_2.$$

By part 3, $F(s, x_t) \notin q_0 + H$ for all $t \le 0$ with $x_t \in N_2 \cap (q_0 + H) \subset N_1 \cap (q_0 + H)$ and all $s \in (0, \epsilon_1]$. For the same t and for all $s \in (t_q - \epsilon, \tau(x_t))$ we have

$$F(s, x_t) \notin q_0 + H$$

due to the properties of τ. Now the assertion becomes obvious.

5. Choose an open neighbourhood $N_3 \subset N_2$ of q_0 in C with

$$P(N_3 \cap (q_0 + H)) \subset N_0.$$

Claim: There is an open neighbourhood $N_{\mathcal{Q}}$ of $\mathcal{Q}$ with

$$N_{\mathcal{Q}} \cap (N_0 \cap (q_0 + H)) \subset N_3.$$

Proof: Part 2 permits to find open neighbourhoods N_t of q_t, $0 < t < t_q$, so that

$$N_t \cap (\overline{N_0} \cap (q_0 + H)) = \emptyset.$$

Set

$$N_{\mathcal{Q}} = N_3 \cup \left(\cup_{0<t<t_q} N_t\right).$$

Then each $\phi \in N_{\mathcal{Q}} \cap (N_0 \cap (q_0 + H))$ satisfies $\phi \notin N_t$ for all $t \in (0, t_q)$; using $\phi \in N_{\mathcal{Q}}$ we infer $\phi \in N_3$.

6. Part 1 permits to find $t_{\mathcal{Q}} \leq 0$ with

$$x_t \in N_{\mathcal{Q}} \qquad \text{for all } t \leq t_{\mathcal{Q}}.$$

Claim: The set

$$T = \{t \leq t_{\mathcal{Q}} : x_t \in (N_0 \cap (q_0 + H))\}$$

is nonempty and not bounded from below.

Proof: The hypothesis $\alpha(x) = \mathcal{Q}$, boundedness of $x|_{(-\infty,0]}$, Eq. (1.1), and the Arzèla–Ascoli theorem combined yield a sequence $(t_j)_{-\infty}^0$ in $(-\infty, 0]$ and $t \in [0, t_q]$ with $t_j \to -\infty$ and $x_{t_j} \to q_t$ as $j \to -\infty$. It follows that

$$x_{t_j+t_q-t} \to q_{t+t_q-t} = q_0 \qquad \text{as } j \to -\infty;$$

there exists $j_0 \in -\mathbb{N}$ with $t_j + t_q - t \leq t_{\mathcal{Q}}$, $x_{t_j+t_q-t} \in N_0$, and

$$F(\tau(x_{t_j+t_q-t}), x_{t_j+t_q-t}) \in N_0 \cap (q_0 + H)$$

for all integers $j \leq j_0$.

7. Recall $N_{\mathcal{Q}} \cap (N_0 \cap (q_0 + H)) \subset N_3 \subset N_1$. Part 3 shows that the set T is discrete. Using part 6 we obtain that T consists of the values of a strictly increasing sequence $(t_j)_{-\infty}^0$ in $(-\infty, t_{\mathcal{Q}}]$, with $t_j \to -\infty$ as $j \to -\infty$.

Claim: For every $j \in -\mathbb{N}$ with $j \leq -1$,

$$t_{j+1} = t_j + \tau(x_{t_j}).$$

Proof: Let $j \in -\mathbb{N}$, $j \leq -1$. $t_j \in T$ gives $x_{t_j} \in N_{\mathcal{Q}} \cap (N_0 \cap (q_0 + H)) \subset N_3 \subset N_2$. The assumption

$$F(s, x_{t_j}) \in N_{\mathcal{Q}} \cap (N_0 \cap (q_0 + H)) \qquad \text{for some } s \in (0, \tau(x_{t_j}))$$

implies $F(s, x_{t_j}) \in N_3 \cap (q_0 + H)$, according to the claim in part 5, in contradiction to $N_3 \subset N_2$ and to the result of part 4. It follows that

$$t_j + \tau(x_{t_j}) \leq t_{j+1}.$$

Observe that $t = t_j + \tau(x_{t_j})$ satisfies $t \leq t_{j+1} \leq t_{\mathcal{Q}}$, $x_t \in q_0 + H$, and due to $x_{t_j} \in N_3$ and to part 5, $x_t \in N_0$. Altogether, we obtain $t \leq t_{\mathcal{Q}}$ and

$$x_t \in N_0 \cap (q_0 + H),$$

which gives $t \in T$. Using $t \leq t_{j+1}$ we infer $t = t_{j+1}$.

8. It follows that

$$x_{t_{j+1}} = P(x_{t_j}) \qquad \text{for all } j \in -\mathbb{N} \text{ with } j \leq -1.$$

Every subsequence of $(x_{t_j})_{-\infty}^{0}$ has a further subsequence which converges to some $q_t \in \mathcal{Q} \cap (\overline{N_0} \cap (q_0 + H)) = \{q_0\}$. This implies

$$x_{t_j} \to q_0 \qquad \text{as } j \to -\infty.$$

$\square$

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