Global stability for price models with delay*

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Abstract

Consider the delay differential equation $\dot{x}(t) = a\int_0^r x(t-s)\,d\eta(s) - g(x(t))$ and the neutral type differential equation $\dot{y}(t) = a\int_0^r \dot{y}(t-s)\,d\mu(s) - g(y(t))$ where $a > 0$, $g : \mathbb{R} \to \mathbb{R}$ is smooth, $ug(u) > 0$ for $u \neq 0$, $\int_0^s g(u)\,du \to \infty$ as $|s| \to \infty$, $r > 0$, η and μ are nonnegative functions of bounded variation on $[0, r]$, $\eta(0) = \eta(r) = 0$, $\int_0^r \eta(s)\,ds = 1$, μ is nondecreasing, μ does not have a singular part, $\int_0^r d\mu = 1$. Both equations can be interpreted as price models. Global asymptotic stability of $y = 0$ is obtained, in case $a \in (0, 1)$, for the neutral equation by using a Lyapunov functional. Then this result is applied to get global asymptotic stability of $x = 0$ for the (non-neutral) delay differential equation provided $a \in (0, 1)$. As particular cases, two related global stability conjectures are solved, with an affirmative answer.

Keywords: neutral differential equation, delay differential equation, price model, global asymptotic stability

1 Introduction

Our primary aim is to prove the global stability conjecture for the price model of Erdélyi, Brunovský and Walther [1, 2, 9]

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t), \tag{1.1}$$

where $a > 0, \beta > 0$. For $0 < a < 1$, the local asymptotic stability of $x = 0$ was shown by Erdélyi, Brunovský, and Walther, and they conjectured global asymptotic stability. Numerical simulations provided by Erdélyi [3] suggested the existence of a stable (slowly oscillating) periodic solution of Eq. (1.1) for $a > 1$, which was established in [1, 2]. This result has recently been generalized by Stumpf [8] for a state-dependent delay version of Eq. (1.1). Walther analyzed further the slowly oscillating periodic solution of Eq. (1.1) and showed that it converges to a square-wave solution as a tends to infinity [9], and that the period tends to infinity as $a \to 1^+$ [10].

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Recently, Garab, Kovács and Krisztin [4] obtained global asymptotic stability of $x = 0$ for Eq. (1.1) provided $a \in (0, 0.61)$. The key idea of [4] to prove global asymptotic stability of $x = 0$ was to rewrite the equation as a neutral type functional differential equation. Then an equivalent equation with infinite delay was obtained for which a stability result of [7] was applied. The technique of [4] worked for the more general price model

$$\dot{x}(t) = a \sum_{i=1}^{n} b_i[x(t - s_i) - x(t - r_i)] - g(x(t)), \tag{1.2}$$

as well, where $a > 0$, $b_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, $\sum_{i=1}^{n} b_i(r_i - s_i) = 1$ holds, and g is a smooth increasing real function with $ug(u) > 0$ for $u \ne 0$. [4] proved global asymptotic stability for Eq. (1.2) when $a \in (0, 1)$ and an additional condition was assumed, see the details in Sect. 3. In [4] it remained open to prove global asymptotic stability without the additional condition, i.e., for $a \in (0, 1)$.

In the sequel, we always assume $r > 0$, $a > 0$, and

$$(\mathrm{H_g}) \quad \begin{cases} g : \mathbb{R} \to \mathbb{R} \text{ is } C^1\text{−smooth, } ug(u) > 0 \text{ for } u \ne 0, \\ \int_0^s g(u)\, du \to \infty \text{ as } |s| \to \infty. \end{cases}$$

By using Stieltjes integrals, Eqs. (1.1) and (1.2) can be written as

$$\dot{x}(t) = a \int_0^r x(t - s) d\eta(s) - g(x(t)) \tag{1.3}$$

with η satisfying

$$(\mathrm{H}_\eta) \quad \begin{cases} \eta : [0, r] \to [0, \infty) \text{ is of bounded variation,} \\ \eta(0) = \eta(r) = 0, \ \int_0^r \eta(s)\, ds = 1. \end{cases}$$

Following [1], $x(t)$ in Eq. (1.3) can represent the price of an asset at time t. Indeed, if $x : I \to \mathbb{R}$ is continuously differentiable on an interval containing $[t-r, t]$, then integrating the Stieltjes integral $\int_0^r x(t - s)\, d\eta(s)$ by parts, and using $\eta(0) = \eta(r) = 0$, we find

$$\begin{aligned} \int_0^r x(t - s)\, d\eta(s) &= [x(t - s)\eta(s)]_{s=0}^{s=r} - \int_0^r \eta(s)\, d_s x(t - s) \\ &= -\int_0^r \eta(s) \tfrac{d}{ds} x(t - s)\, ds \\ &= \int_0^r \dot{x}(t - s)\, d_s \left(\int_0^s \eta\right). \end{aligned} \tag{1.4}$$

As η is nonnegative, the function $[0, r] \ni s \mapsto \int_0^s \eta \in \mathbb{R}$ is monotone nondecreasing. Then (1.4) shows that the term $\int_0^r x(t - s)\, d\eta(s)$ is zero if x is constant on $[t - r, t]$, and it is positive (negative) if $\dot{x}(s) > 0$ (< 0) for all $s \in [t - r, t]$. Therefore, the term $\int_0^r x(t - s)\, d\eta(s)$ can be used to describe the tendency of the price, and the term $a \int_0^r x(t - s)\, d\eta(s)$ with $a > 0$ can represent the positive response to the recent tendency of the price. The term $-g(x(t))$ in Eq. (1.3) is responsible for the negative feedback to the deviation of the price from the zero equilibrium.

Observe that if the function $s \mapsto \int_0^s \eta(u) du$ in the integral term $\int_0^r \dot{x}(t - s)\, d_s \left(\int_0^s \eta\right)$ in equality (1.4) is replaced by an arbitrary nondecreasing function $\mu : [0, r] \to \mathbb{R}$ of bounded variation, then the obtained integral term $\int_0^r \dot{x}(t - s)\, d\mu(s)$ can be still interpreted as the tendency of the price. This motivates to study the neutral type differential equation

$$\dot{y}(t) = a \int_0^r \dot{y}(t - s) d\mu(s) - g(y(t)), \tag{1.5}$$

as well as a price model provided $a > 0$ and $\mu : [0, r] \to \mathbb{R}$ is of bounded variation and nondecreasing with an additional technical assumption given in Sect. 2.

There is another reason to study the neutral type eqaution (1.5). It plays a crucial role in the proof of the stability results for Eqs. (1.1), (1.2), (1.3). However, Eq. (1.3) and Eq. (1.5) are not equivalent. A solution of Eq. (1.3) satisfies Eq. (1.5) with $\mu(s) = \int_0^s \eta$ only for $t > r$. The phase spaces and the stability definitions are also different for Eqs. (1.3) and (1.5).

The paper is organized as follows. In Sect. 2, we consider Eq. (1.5), formulate the hypotheses on μ, and introduce a suitable phase space. First it is shown that all solutions can be globally extended to $[-r, \infty)$. Then a sufficient condition is given for the global asymptotic stability of the zero solution of Eq. (1.5). The proof is based on a Lyapunov functional which has been inspired by the one employed for the equation

$$\dot{x}(t) = a\dot{x}(t-1) - g(x(t)) \tag{1.6}$$

in the book of Kolmanovskii and Myshkis [6, Chapter 9, p. 374]. There, Eq. (1.6) describes a shunted power transmission line.

In Sect. 3, we consider Eq. (1.3) under hypotheses $(\mathrm{H_g})$ and (H_η). Combining the global stability result of Sect. 2 for Eq. (1.5) and the continuous dependence on initial data for Eq. (1.3), the main result is that the zero solution of Eq. (1.3) is globally asymptotically stable provided $a \in (0, 1)$. As a consequence, global asymptotic stability is obtained for the zero solution of the Erdélyi–Brunovský–Walther equation (1.1) and also for Eq. (1.2) for the full conjectured region $a \in (0, 1)$.

Finally in Sect. 4 we show that the global stability result for Eq. (1.3) is optimal in the sense that for $a > 1$ under the additional condition $g'(0) = 0$ the zero solution is unstable. In addition, some open problems are mentioned.

2 Global stability in Eq. (1.5)

In this section we study Eq. (1.5) under condition $a \in (0, 1)$, hypothesis $(\mathrm{H_g})$, and the assumption on μ descibed below.

First we define a step function with (possibly) infinite number of steps. Let $(c_n)_{n=0}^\infty$ be a sequence of nonnegative numbers with $\sum_{n=0}^\infty c_n \le 1$, and let $(r_n)_{n=0}^\infty$ be a sequence in $[0, r]$ such that $r_0 = 0$, and $r_n > 0$ for all $n \in \mathbb{N}$. Let $H : [0, r] \to \mathbb{R}$ be given by $H(0) = 0$, $H(s) = 1$ for $s \in (0, r]$. Define $\sigma : [0, r] \to \mathbb{R}$ by

$$\sigma(s) = c_0 H(s) + \sum_{n:\ r_n \le s} c_n, \qquad s \in [-r, 0].$$

Let a nondecreasing and absolutely continuous $\nu : [0, r] \to \mathbb{R}$ be given with $\nu(r) - \nu(s) \le 1$.

Our hypothesis on μ is that it is nondecreasing without a singular part, that is,

$$(\mathrm{H}_\mu) \quad \begin{cases} \mu : [0, r] \to \mathbb{R} \text{ is given by } \mu = \nu + \sigma \\ \text{such that } \int_0^r d\mu = 1, \text{ i.e., } \nu(r) - \nu(0) + \sum_{n=0}^\infty c_n = 1 \end{cases}$$

holds.

For a continuous function $u : I \to \mathbb{R}$ on an interval I with $t - r, t \in I$ define $u_t \in C([-r, 0], \mathbb{R})$ by $u_t(s) = u(t + s)$, $-r \le s \le 0$. For $\varphi \in C([-r, 0], \mathbb{R})$ set $||\varphi|| = \max_{-r \le s \le 0} |\varphi(s)|$.

Define the set

$$Y = \left\{ \psi \in C^1([-r,0],\mathbb{R}) : \ \dot{\psi}(0) = a \int_0^r \dot{\psi}(-s)\, d\mu(s) - g(\psi(0)) \right\},$$

and let

$$||\psi||_Y = \left((\psi(0))^2 + \int_0^r (\dot{\psi}(-s))^2\, ds \right)^{1/2}$$

for $\psi \in Y$. Set Y will be the phase space for Eq. (1.5).

A solution of Eq. (1.5) with initial function $\psi \in Y$ is a continuously differentiable function $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$ such that $y_0 = \psi$, and Eq. (1.5) holds for all $t \in (0, t_\psi)$. The solution y^ψ is called a maximal solution if any other solution with the same initial function is a restriction of y^ψ.

From $g(0) = 0$ it is clear that $y = 0$ is a solution of (1.5), and by ($\mathrm{H_g}$) it is the only equilibrium solution. The solution $y = 0$ of Eq. (1.5) is called stable if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for each $\psi \in Y$ with $||\psi||_Y < \delta(\epsilon)$, the solution y^ψ exists on $[-r,\infty)$ and $||y_t^\psi||_Y < \epsilon$ for all $t \geq 0$. The solution $y = 0$ is called globally asymptotically stable if it is stable and for each $\psi \in Y$ the solution y^ψ exists on $[-r,\infty)$ and $||y_t^\psi||_Y \to 0$ as $t \to \infty$.

Theorem 3.1 on page 107 of Kolmanovskii and Myshkis [6] states that for each $\psi \in Y$, Eq. (1.5) has a unique maximal solution $y^\psi : [-r, t_\psi) \to \mathbb{R}$, and in case $t_\psi < \infty$ the finite limit $\lim_{t\to t_\psi -} \dot{y}^\psi(t)$ does not exist. We will use this result to show that for any $\psi \in Y$ there exists a unique solution on $[-r,\infty)$.

Proposition 2.1. *Assume hypotheses* ($\mathrm{H_g}$), (H_μ) *hold, and* $a \in (0,1)$. *Let* $\psi \in Y$ *and consider the unique maximal solution* $y^\psi : [-r, t_\psi) \to \mathbb{R}$ *of Eq. (1.5). If* y^ψ *is bounded on* $[-r, t_\psi)$ *then* $t_\psi = \infty$.

Proof. Let $\psi \in Y$, $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$, and let y be bounded on $[-r, t_\psi)$.

Assume $t_\psi < \infty$. Then, by [6, Theorem 3.1 on p. 107], the finite limit $\lim_{t\to t_\psi -} \dot{y}(t)$ does not exist.

First we show that $\dot{y}$ is bounded on $[-r, t_\psi)$. If $\dot{y}$ is unbounded from above on $[-r, t_\psi)$ then we can choose a sequence $(\tau_n)_{n=1}^\infty$ in $[0, t_\psi)$ such that $\tau_n \to t_\psi$, $\dot{y}(\tau_n) \to \infty$ as $n \to \infty$, $\dot{y}(t) < \dot{y}(\tau_n)$ for all $t \in [-r, \tau_n)$, $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$, by using hypothesis (H_μ), we have

$$\begin{aligned} \dot{y}(\tau_n) &= a \int_0^r \dot{y}(\tau_n - s) d\mu(s) - g(y(\tau_n)) \\ &\leq a \int_0^r \dot{y}(\tau_n) d\mu(s) - g(y(\tau_n)) = a\dot{y}(\tau_n) - g(y(\tau(t_n))). \end{aligned}$$

Hence, from $a \in (0,1)$ and $\dot{y}(\tau_n) \to \infty$, it follows that

$$-g(y(\tau_n)) \geq (1-a)\dot{y}(\tau_n) \to \infty \quad \text{as} \quad n \to \infty.$$

As y is bounded, this is a contradiction. The case when $\dot{y}$ is unbounded from below leads similarly to a contradiction. Thus, $\dot{y}$ is bounded on $[-r, t_\psi)$.

Define

$$\alpha = \liminf_{t\to t_\psi^-} \dot{y}(t), \quad \beta = \limsup_{t\to t_\psi^-} \dot{y}(t).$$

We know that $-\infty < \alpha < \beta < \infty$. There are strictly increasing sequences $(t_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty}$ in $[0, t_\psi)$ such that $t_n \to t_\psi$, $s_n \to t_\psi$, and

$$\lim_{n\to\infty} \dot{y}(s_n) = \alpha, \quad \lim_{n\to\infty} \dot{y}(t_n) = \beta.$$

Choose $\alpha' < \alpha < \beta < \beta'$ so that $a(\beta' - \alpha') < \beta - \alpha$. There exists a $\delta > 0$ such that $\dot{y}(t) \in [\alpha', \beta']$ for all $t \in [t_\psi - 2\delta, t_\psi)$. From (1.5) it follows that

$$\begin{aligned} \dot{y}(t_n) - \dot{y}(s_n) &= a\int_0^\delta (\dot{y}(t_n - s) - \dot{y}(s_n - s))d\mu(s) \\ &\quad + a\int_\delta^r (\dot{y}(t_n - s) - \dot{y}(s_n - s))d\mu(s) - g(y(t_n)) + g(y(s_n)). \end{aligned}$$

We have

$$\lim_{n\to\infty} a\int_\delta^r (\dot{y}(t_n - s) - \dot{y}(s_n - s))d\mu(s) = 0$$

because $\dot{y}$ is uniformly continuous on $[-r, t_\psi - \delta]$. In addition,

$$\lim_{n\to\infty} [g(y(t_n)) - g(y(s_n))] = 0$$

since the boundedness of $\dot{y}$ on $[-r, t_\psi)$ implies the uniform continuity of y and $g \circ y$, and that $g(y(t))$ has a finite limit at t_ψ. Combining these facts with $\dot{y}(t_n) - \dot{y}(s_n) \to \beta - \alpha$ as $n \to \infty$, one obtains

$$\begin{aligned} \beta - \alpha &= \lim_{n\to\infty} (\dot{y}(t_n) - \dot{y}(s_n)) \\ &= \lim_{n\to\infty} a\int_0^\delta (\dot{y}(t_n - s) - \dot{y}(s_n - s))\, d\mu(s) \\ &\le a\int_0^\delta (\beta' - \alpha')\, d\mu(s) \\ &\le a(\beta' - \alpha'), \end{aligned}$$

a contradiction. Therefore, $t_\psi = \infty$, and the proof is complete. □

Next we show the global asymptotic stability of the zero solution of Eq. (1.5).

Theorem 2.2. *Assume hypotheses* ($\mathrm{H_g}$), (H_μ) *hold, and* $a \in (0, 1)$. *Then for each* $\psi \in Y$ *the unique maximal solution* y^ψ *of Eq. (1.5) is defined on* $[-r, \infty)$, *and the zero solution of (1.5) is globally asymptotically stable.*

Proof. Define the function

$$K : [0, r] \ni s \mapsto \int_s^r d\mu \in [0, 1].$$

According to hypothesis (H_μ), let $K_1(s) = \int_s^r d\nu$ and $K_2(s) = \int_s^r d\sigma$. Then $K(s) = K_1(s) + K_2(s)$, and

$$K_1(s) = \nu(r) - \nu(s), \quad K_2(s) = \begin{cases} \sum_{n=0}^{\infty} c_n & \text{for } s = 0, \\ \sum_{r_n > s} c_n & \text{for } s \in (0, r]. \end{cases}$$

Let $\psi \in Y$ and consider the unique solution $y = y^{\psi} : [-r, t_{\psi}) \to \mathbb{R}$. For $t \in [0, t_{\psi})$, define

$$\begin{aligned} w(t) &= \int_{t-r}^{t} K(t-s)(\dot{y}(s))^2\, ds + \frac{2}{a^2}\int_0^{y(t)} g(u)\, du \\ &= \int_{t-r}^{t} \left[K_1(t-s) + K_2(t-s)\right](\dot{y}(s))^2\, ds + \frac{2}{a^2}\int_0^{y(t)} g(u)\, du. \end{aligned}$$

As y and K_1 are continuously differentiable functions, the map

$$[0, t_{\psi}) \ni t \mapsto \int_{t-r}^{t} K_1(t-s)(\dot{y}(s))^2\, ds \in \mathbb{R}$$

is continuously differentiable, and

$$\begin{aligned} &\tfrac{d}{dt} \int_{t-r}^{t} K_1(t-s)(\dot{y}(s))^2\, ds \\ &= K_1(0)(\dot{y}(t))^2 - K_1(r)(\dot{y}(t-r))^2 + \int_{t-r}^{t} \frac{d}{dt} K_1(t-s)(\dot{y}(s))^2\, ds \\ &= \left[\nu(r) - \nu(0)\right](\dot{y}(t))^2 + \int_0^r (\dot{y}(t-s))^2 K_1'(s)\, ds \\ &= \left[\nu(r) - \nu(0)\right](\dot{y}(t))^2 - \int_0^r (\dot{y}(t-s))^2\, d\nu(s). \end{aligned}$$

Observe that

$$\int_{t-r}^{t} K_2(t-s)(\dot{y}(s))^2\, ds = \sum_{n=0}^{\infty} c_n \int_{t-r_n}^{t} (\dot{y}(s))^2\, ds.$$

This series of functions is continuously differentiable, it can be differentiated term by term, and

$$\begin{aligned} \frac{d}{dt}\int_{t-r}^{t} K_2(t-s)(\dot{y}(s))^2\, ds &= \sum_{n=0}^{\infty} c_n \left[(\dot{y}(t))^2 - (\dot{y}(t-r_n))^2\right] \\ &= \left(\sum_{n=0}^{\infty} c_n\right)(\dot{y}(t))^2 - \int_0^r (\dot{y}(t-s))^2\, d\sigma(s). \end{aligned}$$

The last term in $w(t)$ is clearly continuously differentiable with

$$\frac{d}{dt}\frac{2}{a^2}\int_0^{y(t)} g(u)\, du = \frac{2}{a^2} g(y(t))\dot{y}(t).$$

Therefore, w is continuously differentiable on $[0, t_{\psi})$, and

$$w'(t) = \left[\nu(r) - \nu(0) + \sum_{n=0}^{\infty} c_n\right](\dot{y}(t))^2 - \int_0^r (\dot{y}(t-s))^2\, d(\nu+\sigma)(s) + \frac{2}{a^2} g(y(t))\dot{y}(t).$$

By hypothesis (H_{μ}), we have $\nu(r) - \nu(0) + \sum_{n=0}^{\infty} c_n = 1$, and $\mu = \nu + \sigma$. Jensen's inequality implies

$$\left(\int_0^r \dot{y}(t-s)\, d\mu(s)\right)^2 \le \int_0^r (\dot{y}(t-s))^2\, d\mu(s).$$

Combining the above relations, it follows that

$$w'(t) \leq (\dot{y}(t))^2 - \left(\int_0^r \dot{y}(t-s)\, d\mu(s) \right)^2 + \frac{2}{a^2} g(y(t))\dot{y}(t). \tag{2.1}$$

From Eq. (1.5), the term $\int_0^r \dot{y}(t-s)\, d\mu(s)$ is equal to $(1/a)[\dot{y}(t) + g(y(t))]$. Therefore, by (2.1),

$$w'(t) \leq -\left(\frac{1}{a^2} - 1 \right) (\dot{y}(t))^2 - \frac{1}{a^2}(g(y(t)))^2 \tag{2.2}$$

holds for all $t \in [0, t_\psi)$.

From inequality (2.2), by $a \in (0,1)$, it follows that w is a nonincreasing function on $[0, t_\psi)$, and $w(t) \in [0, w(0)]$ for all $t \in [0, t_\psi)$. This fact and the definition of w gives

$$\int_0^{y(t)} g(u)\, du \in \left[0, \frac{a^2}{2} w(0) \right]$$

for all $t \in [0, t_\psi)$. By hypothesis ($\mathrm{H_g}$) we obtain that y is bounded on $[0, t_\psi)$, and then on $[-r, t_\psi)$. Proposition 2.1 can be applied to conclude $t_\psi = \infty$.

Thus, inequality (2.2) holds for all $t \in [0, \infty)$. Then there exists $w_* \geq 0$ such that $w(t) \to w_*$ as $t \to \infty$, and, for each $T \geq 0$,

$$\begin{aligned} w(0) - w_* \geq w(0) - w(T) &= -\int_0^T w'(t)\, dt \\ &\geq \left(\frac{1}{a^2} - 1 \right) \int_0^T (\dot{y}(t))^2\, dt + \frac{1}{a^2} \int_0^T (g(y(t)))^2\, dt. \end{aligned}$$

Hence it follows that

$$\int_0^\infty (\dot{y}(t))^2\, dt \leq \frac{a^2}{1-a^2} w(0), \tag{2.3}$$

$$\int_0^\infty (g(y(t)))^2\, dt \leq a^2 w(0). \tag{2.4}$$

In particular, (2.3) implies

$$\int_{t-r}^t (\dot{y}(s))^2\, ds \to 0 \quad \text{as } t \to \infty.$$

Then, using $K(s) \in [0,1]$, $s \in [0, r]$, one finds that

$$\int_0^r K(s)(\dot{y}(t-s))^2\, ds \leq \int_0^r (\dot{y}(t-s))^2\, ds = \int_{t-r}^t (\dot{y}(s))^2\, ds,$$

and

$$\int_{t-r}^t K(t-s)(\dot{y}(s))^2\, ds = \int_0^r K(s)(\dot{y}(t-s))^2\, ds \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore,

$$w_* = \lim_{t\to\infty} w(t) = \frac{2}{a^2} \lim_{t\to\infty} \int_0^{y(t)} g(u)\, du.$$

From condition ($\mathrm{H_g}$), the map $[0,\infty) \ni s \mapsto \int_0^s g(u)\,du \in \mathbb{R}$ strictly increases from 0 to ∞, and the map $(-\infty,0] \ni s \mapsto \int_0^s g(u)\,du \in \mathbb{R}$ strictly decreases from ∞ to 0. Consequently, there exists $y_* \in \mathbb{R}$ so that $y(t) \to y_*$ as $t \to \infty$. By (2.4), the integral $\int_0^\infty (g(y(t)))^2\,dt$ converges. These facts combined yield $y_* = 0$. Thus, $y^\psi(t) \to 0$ as $t \to \infty$ for all $\psi \in Y$.

In order to show local stability of the zero solution, let $\epsilon >$ be given. By hypothesis ($\mathrm{H_g}$), there exists $m \geq a^2$ such that

$$|g(u)| \leq m|u| \quad \text{for all} \quad u \in [-1,1].$$

Choose $\delta = \delta(\epsilon) \in (0,1)$ so that

$$\left(1 + \frac{m}{1-a^2}\right)\delta^2 < \frac{\epsilon^2}{2}. \tag{2.5}$$

In addition, by ($\mathrm{H_g}$), δ can be chosen so small that

$$\int_0^s g(u)\,du < \frac{m}{2}\delta^2 \quad \text{implies} \quad s^2 < \frac{\epsilon^2}{2}. \tag{2.6}$$

Let $\psi \in Y$ with $||\psi||_Y < \delta$, and let $y = y^\psi : [-r,\infty) \to \mathbb{R}$ be the corresponding solution of (1.5). Then $K(s) \in [0,1]$, $s \in [0,r]$, $|\psi(0)| \leq ||\psi||_Y < \delta < 1$, and the choice of m guarantee that

$$\begin{aligned} w(0) &= \int_{-r}^0 K(-s)(\dot\psi(s))^2\,ds + \frac{2}{a^2}\int_0^{\psi(0)} g(u)\,du \\ &\leq \int_{-r}^0 (\dot\psi(s))^2\,ds + \frac{2}{a^2}\frac{m}{2}(\psi(0))^2 \\ &\leq \frac{m}{a^2}\left(\int_{-r}^0 (\dot\psi(s))^2\,ds + (\psi(0))^2\right) \\ &= \frac{m}{a^2}||\psi||_Y^2 < \frac{m}{a^2}\delta^2. \end{aligned}$$

This estimation for $w(0)$ combined with inequality (2.3) yields

$$\int_0^\infty (\dot y(t))^2\,dt < \frac{m}{1-a^2}\delta^2. \tag{2.7}$$

The estimation $w(0) < (m/a^2)\delta^2$, the definition of $w(t)$, and $w(t) \leq w(0)$ combined give

$$\int_0^{y(t)} g(u)\,du \leq \frac{a^2}{2}w(t) \leq \frac{a^2}{2}w(0) < \frac{m}{2}\delta^2, \quad t \geq 0. \tag{2.8}$$

From $||\psi||_Y < \delta$ and inequality (2.7) it follows that

$$\int_{-r}^0 (\dot y(t+s))^2\,ds < \left(1 + \frac{m}{1-a^2}\right)\delta^2 \quad \text{for all} \quad t \geq 0. \tag{2.9}$$

A combination of (2.9), (2.5), (2.6), (2.8) implies

$$||y_t||_Y = \left(\int_{-r}^0 (\dot y(t+s))^2\,ds + (y(t))^2\right)^{1/2} < \left(\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}\right)^{1/2} = \epsilon$$

for all $t \geq 0$. This proves the local stability of $y = 0$.

The global attractivity of $y = 0$, that is $||y_t^\psi||_Y \to 0$ as $t \to \infty$, for all $\psi \in Y$, follows from $\int_{-r}^0 (\dot y^\psi(t+s))^2\,ds \to 0$, $t \to \infty$, implied by (2.3), and $y^\psi(t) \to 0$, $t \to \infty$. Therefore, $y = 0$ is globally asymptotically stable. $\square$

3 Global stability in Eq. (1.3)

In this section, we consider Eq. (1.3) under hypotheses $(\mathrm{H_g})$ and (H_η).

The natural phase space for Eq. (1.3) is $C([-r,0],\mathbb{R})$. A maximal solution of (1.3) with initial function $\varphi \in C([-r,0],\mathbb{R})$ is a continuous function $x = x^\varphi : [-r, t_\varphi) \to \mathbb{R}$ with $t_\varphi > 0$ so that $x|_{[-r,0]} = \varphi$, x is differentiable on $(0, t_\varphi)$, Eq. (1.3) holds on $(0, t_\varphi)$, and any other solution with the same initial function is a restriction of x^φ.

Recall that the solution $x = 0$ of Eq. (1.3) is stable if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for each $\varphi \in C([-r,0],\mathbb{R})$ with $||\varphi|| < \delta(\epsilon)$, the solution x^φ exists on $[-r,\infty)$ and $||x_t^\varphi|| < \epsilon$ for all $t \geq 0$. The solution $x = 0$ is globally asymptotically stable if in addition to stability for each $\varphi \in C([-r,0],\mathbb{R})$ the solution x^φ exists on $[-r,\infty)$ and $||x_t^\varphi|| \to 0$ as $t \to \infty$.

First, for arbitrary $a > 0$, we show that the maximal solutions exist on $[-r,\infty)$.

Proposition 3.1. *Assume that $a > 0$ and conditions $(\mathrm{H_g})$, (H_η) hold. For each $\varphi \in C([-r,0],\mathbb{R})$ Eq. (1.3) has a unique maximal solution x^φ which is defined on $[-r,\infty)$.*

Proof. The map

$$f : C([-r,0],\mathbb{R}) \ni \varphi \mapsto a \int_0^r \varphi(-s)\, d\eta(s) - g(\varphi(0)) \in \mathbb{R}$$

is continuous, it is also Lipschitzian in each compact subset of $C([-r,0],\mathbb{R})$, and f takes bounded sets into bounded sets. Then, by [5, Chapter 2, Theorem 2.3], for each $\varphi \in C([-r,0],\mathbb{R})$ there is a unique maximal solution $x^\varphi : [-r, t_\varphi) \to \mathbb{R}$ of Eq. (1.3). Moreover, by [5, Chapter 2, Theorem 3.2], in case $t_\varphi < \infty$ we have $||x_t^\varphi|| \to \infty$ as $t \to t_\varphi^-$.

Let $\varphi \in C([-r,0],\mathbb{R})$, and let $|\eta|$ denote the total variation of η. Define

$$k(t) = (||\varphi|| + 1)\, e^{(a|\eta|+1)t}, \quad t \in [0,\infty).$$

We claim that

$$|x^\varphi(t)| < k(t) \quad \text{for all} \quad t \in [0, t_\varphi). \tag{3.1}$$

If inequality (3.1) does not hold then, by $|x^\varphi(0)| \leq ||\varphi|| < k(0)$, there exists $t_0 \in (0, t_\varphi)$ such that $|x^\varphi(t)| < k(t)$ for all $t \in [0, t_0)$, $|x^\varphi(t_0)| = k(t_0)$, $|\dot{x}^\varphi(t_0)| \geq k'(t_0) = (a|\eta| + 1)k(t_0)$. Assume $x^\varphi(t_0) = k(t_0)$ (the case $-x^\varphi(t_0) = k(t_0)$ is similar). Then Eq. (1.3), $k(t) < k(t_0)$ for $t \in [0, t_0)$, and $g(x^\varphi(t_0)) > 0$ combined yield the contradiction

$$\dot{x}^\varphi(t_0) \leq a|\eta|k(t_0) - g(x^\varphi(t_0)) < a|\eta|k(t_0).$$

Therefore, inequality (3.1) holds. Then, by [5, Chapter 2, Theorem 3.2], $t_\varphi = \infty$. $\square$

Now we consider Eq. (1.3) for $a \in (0,1)$, and prove the global asymptotic stability of the zero solution.

Theorem 3.2. *Assume hypotheses $(\mathrm{H_g})$, (H_η) hold, and $a \in (0,1)$. Then the zero solution of Eq. (1.3) is globally asymptotically stable.*

Proof. In order to show local stability, let $\epsilon > 0$ be given. By Theorem 2.2 there exists $\gamma = \gamma(\epsilon) > 0$ such that for each $\psi \in Y$ with $||\psi||_Y < \gamma$, for the solution y^ψ of Eq. (1.5), the inequality $||y_t^\psi||_Y < \epsilon$ holds for all $t \geq 0$.

Define

$$\delta_1 = \frac{1}{\sqrt{1+r}}\gamma.$$

Let $|\eta|$ denote the total variation of η. By condition ($\mathrm{H_g}$), we can find $\delta_2 \in (0, \delta_1)$ such that

$$(1 + a|\eta|)\delta_2 + \max_{|u|\le\delta_2} |g(u)| < \delta_1.$$

By continuous dependence on initial data of solutions of Eq. (1.3), see [5, Chapter 2, Theorem 2.2], we can choose $\delta > 0$ such that, for each $\varphi \in C([-r, 0], \mathbb{R})$ with $||\varphi|| < \delta$, the unique solution x^φ of Eq. (1.3) satisfies

$$|x^\varphi(t)| < \min\{\epsilon, \delta_2\} \quad \text{for all} \quad t \in [-r, r].$$

Then for x^φ with $\varphi \in C([-r, 0], \mathbb{R})$ and $||\varphi|| < \delta$, from Eq. (1.3) it follows that

$$|\dot{x}^\varphi(t)| \le a|\eta|\delta_2 + \max_{|u|\le\delta_2} |g(u)| < \delta_1 \quad \text{for all} \quad t \in (0, r]$$

and

$$|x^\varphi(t)| < \delta_2 < \delta_1 \quad \text{for all} \quad t \in [0, r].$$

By the uniform continuity of $x^\varphi|_{[-r,r]}$, the limit $\lim_{t\to 0^+} \dot{x}^\varphi(t)$ exists and equals $a\int_0^r \varphi(-s)\, d\eta(s) - g(\varphi(0))$. It follows that x^φ is right differentiable at $t = 0$, and $x_r^\varphi \in C^1([-r, 0], \mathbb{R})$. Then $x_r^\varphi \in Y$ and

$$||x_r^\varphi||_Y = \left(\int_0^r (\dot{x}^\varphi(t))^2\, dt + (x^\varphi(t))^2\right)^{1/2} < \left(r\delta_1^2 + \delta_1^2\right)^{1/2} = \sqrt{r+1}\delta_1 = \gamma.$$

By (1.4), $y(t) = x^\varphi(t + r)$, $t \in [-r, \infty)$, is a solution of Eq. (1.5) with $\mu(s) = \int_0^s \eta$ and initial function $y_0 = x_r^\varphi \in Y$. Then the choice of γ guarantees that

$$||y_t||_Y = ||x_{t+r}^\varphi||_Y < \epsilon \quad \text{for all} \quad t \ge 0.$$

The definition of $\|\cdot\|_Y$ and the choice of δ imply that for each $\varphi \in C([-r, 0], \mathbb{R})$ with $||\varphi|| < \delta$, the inequality

$$|x^\varphi(t)| < \epsilon \quad \text{for all} \quad t \ge 0$$

holds. Therefore the zero solution of Eq. (1.3) is locally stable.

Global attractivity of $x = 0$ also follows from Theorem 2.2 since $x_r^\varphi \in Y$ for all $\varphi \in C([-r, 0], \mathbb{R})$. □

Eq. (1.1) is a particular case of (1.3) with $r = 1$, $g(u) = \beta|u|u$ and

$$\eta(s) = \begin{cases} 1 & \text{if } s \in (0, 1), \\ 0 & \text{if } s = 0 \text{ or } s = 1. \end{cases}$$

Therefore, Theorem 3.2 implies a solution to the global stability conjecture of [1, 2, 9].

Corollary 3.3. *If $a \in (0, 1)$ then the zero solution of Eq. (1.1) is globally asymptotically stable.*

Let the constants $b_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \dots, n\}$, be given so that $\sum_{i=1}^n b_i(r_i - s_i) = 1$ holds. Define the functions

$$\eta_i(s) = \begin{cases} b_i & \text{if } s \in (s_i, r_i), \\ 0 & \text{if } s \in [0, s_i] \cup [r_i, 1], \end{cases}$$

for $i \in \{1, \ldots, n\}$, and let

$$\eta : [0,1] \ni s \mapsto \sum_{i=1}^{n} \eta_i(s) \in \mathbb{R}.$$

Then it is easy to see that η satisfies condition (H_η) with $r = 1$, and Eq. (1.2) is a particular case of Eq. (1.3). Therefore the following result holds.

Corollary 3.4. *If $a \in (0,1)$, $b_i > 0$, $0 \le s_i < r_i \le 1$, $\sum_{i=1}^{n} b_i(r_i - s_i) = 1$, and g satisfies condition ($\mathrm{H_g}$), then the zero solution of Eq. (1.2) is globally asymptotically stable.*

The equation

$$\dot{x}(t) = \alpha \sum_{i=1}^{n} \beta_i [x(t - s_i) - x(t - r_i)] - g(x(t)), \tag{3.2}$$

was studied in [4] under the conditions $\alpha > 0$, $\beta_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \ldots, n\}$, $\sum_{i=1}^{n} \beta_i = 1$, and g satisfied a condition stronger than ($\mathrm{H_g}$). Setting

$$a = \alpha \sum_{i=1}^{n} \beta_i (r_i - s_i), \qquad b_i = \frac{\beta_i}{\sum_{i=1}^{n} \beta_i (r_i - s_i)} \quad (i \in \{1, \ldots, n\}),$$

it is clear that Eq. (3.2) is equivalent to Eq. (1.2). Consequently, we obtain the following result.

Corollary 3.5. *If $\alpha > 0$, $\beta_i > 0$, $0 \le s_i < r_i \le 1$, $i \in \{1, \ldots, n\}$, and g satisfies ($\mathrm{H_g}$), then*

$$\alpha \sum_{i=1}^{n} \beta_i (r_i - s_i) < 1$$

implies the global asymptotic stability of the zero solution of Eq. (3.2).

We remark that [4] proved global asymptotic stability of $x = 0$ for Eq. (3.2) assuming $\alpha \sum_{i=1}^{n} \beta_i (r_i - s_i) < 1$, a condition on g that is stronger than ($\mathrm{H_g}$), and the extra condition

$$\alpha^2 \sum_{i=1}^{n} \beta_i (r_i^2 - s_i^2) < \left(1 - \alpha \sum_{i=1}^{n} \beta_i (r_i - s_i)\right)^2$$

was also used. By the local stability result for Eq. (3.2), and by the analogous conjecture for Eq. (1.1), it was suspected in [4] that Corollary 3.5 holds.

4 Discussion

In this section we show that the global stability result $a < 1$ for Eq. (1.3), and then also for Eqs. (1.1) and (1.2), is sharp in the sense that under the additional condition $g'(0) = 0$ inequality $a > 1$ implies that the zero solution is unstable. Remark that $g'(0) = 0$ holds for Eq. (1.1). Paper [4] also assumed $g'(0) = 0$ when studied Eq. (3.2) or equivalenty Eq. (1.2).

Theorem 4.1. *Suppose that hypotheses ($\mathrm{H_g}$), (H_η) hold, and $g'(0) = 0$. If $a > 1$ then the zero solution of Eq. (1.3) is unstable.*

Proof. By $g'(0) = 0$, the linear variational equation of Eq. (1.3) is

$$\dot{x}(t) = a \int_0^r x(t-s)\, d\eta(s). \tag{4.1}$$

The characteristic function is $\Delta : \mathbb{C} \ni \lambda \mapsto \lambda - a \int_0^r e^{-\lambda s}\, d\eta(s) \in \mathbb{C}$.

Condition (H_η) and integration by parts for Stieltjes integrals gives

$$\begin{aligned} \int_0^r e^{-\lambda s}\, d\eta(s) &= \left[e^{-\lambda s}\eta(s)\right]_{s=0}^{s=r} - \int_0^r \eta(s)\, d_s\left(e^{-\lambda s}\right) \\ &= \lambda \int_0^r \eta(s) e^{-\lambda s}\, ds. \end{aligned}$$

Hence, if λ is real and tends to ∞, then, using again (H_η), it is clear that

$$\Delta(\lambda) = \lambda - a \int_0^r e^{-\lambda s}\, d\eta(s) = \lambda \left[1 - a \int_0^r \eta(s) e^{-\lambda s}\, ds\right] \to \infty.$$

Combining this fact with $\Delta(0) = -a \int_0^r d\eta = 0$ and $\Delta'(0) = 1 - a \int_0^r \eta(s)\, ds = 1 - a < 0$, it follows that Δ has a real positive zero. Therefore, a classical result, e.g. from [5], yields that $x = 0$ is unstable. $\square$

Eqs. (1.1), (1.2), (1.3) as price models are also important when the zero solution is unstable. In this case there are results about the dynamics only for Eq. (1.1) with a single delay, see [1, 2, 9, 10]. It is an interesting open problem to understand the dynamics in the presence of multiple or distributed delays, that is, for Eqs. (1.2) and (1.3).

We have seen in the Introduction that the neutral differential equation (1.5) is also interesting as a price model. Global asymptotic stability is obtained in this paper for $a \in (0, 1)$ provided ($\mathrm{H_g}$) and (H_μ) hold. However, the understanding of the dynamics is completely open for $a \geq 1$. The simple-looking equation (1.6) is a particular case. There are results only for $a \in (0, 1)$, and the case $a \geq 1$ is also an interesting open problem.

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