Stochastic processes exercises

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Contents

1 Generated σ-algebra, measurability

1.1. Let $(\Omega, \mathcal{A}) = ([0,1], \mathcal{B}([0,1]))$. Let

$$\mathcal{G} = \{A : A \text{ or } A^c \text{ is countable}\}.$$

Show that $\mathcal{G}$ is a σ-algebra, and give a generating system of it.

1.2. Let $(\Omega, \mathcal{A}, \mathbf{P}) = ([0,1], \mathcal{B}_{[0,1]}, \lambda)$, with λ being the Lebesgue measure. Let $\mathcal{F} = \{\emptyset, [0,1/2), [1/2,1], [0,1]\}$. Determine the $\mathcal{F}$-measurable random variables.

Solution. Assume that X is $\mathcal{F}$-measurable. Then for any $a \in \mathbb{R}$, the inverse image $X^{-1}(\{a\}) \in \mathcal{F}$. Thus if $X(\omega) = a$ for some $\omega \in [0,1/2)$, then $X^{-1}(\{a\}) = [0,1/2)$ or $[0,1]$. Indeed, only these two sets in $\mathcal{F}$ contain a point from $[0,1/2)$. Therefore, $X \equiv a$ on $[0,1/2)$. Similarly, X is constant on $[1/2,1]$. Summarizing, a random variable is measurable with respect to $\mathcal{F}$ iff $X(\omega) = aI_{[0,1/2)}(\omega) + bI_{[1/2,1]}(\omega)$, for $a, b \in \mathbb{R}$.

1.3. Let $(\Omega, \mathcal{A})$ be a measurable space. Let $\mathcal{G} = \sigma(B_1, \ldots, B_n)$, where $B_1, \ldots, B_n \in \mathcal{A}$ are disjoint, and $\cup_{i=1}^n B_i = \Omega$. Show that

$$\mathcal{G} = \{\cup_{j \in J} B_j : \ J \subset \{1, 2, \ldots, n\}\}.$$

In particular $|\mathcal{G}| = 2^n$.

1.4. Let $(\Omega, \mathcal{A})$ be a measurable space. What is the structure of a finitely generated σ-algebra $\mathcal{G} = \sigma(A_1, \ldots, A_n)$, with $A_1, \ldots, A_n \in \mathcal{A}$? Characterize the random variables that are measurable with respect $\mathcal{G}$.

1.5. What is the σ-algebra generated by a constant random variable? More generally, what is $\sigma(X)$ if X is discrete random variable?

Solution. Assume that $Y(\omega) \equiv a$. Then for any Borel set B, $Y^{-1}(B) = \emptyset$ or Ω, depending on whether $a \in B$ or $a \notin B$. Thus $\sigma(Y) = \{\emptyset, \Omega\}$, the trivial σ-algebra.

Let Y be a discrete random variable with values $y_1, y_2, \ldots$ (finitely many, or countably infinite). Put $A_i = Y^{-1}(\{y_i\})$. Then $\cup_i A_i = \Omega$, and A_i's are disjoint. Thus $A_i \in \sigma(Y)$, therefore

$$\sigma(A_1, A_2, \ldots) \subset \sigma(Y).$$

Let now B any Borel set, and $j_1, j_2, \ldots$ those indices for which $y_{j_i} \in B$. Then

$$Y^{-1}(B) = Y^{-1}\left(\{y_{j_1}, y_{j_2}, \ldots\}\right) = \cup_i Y^{-1}(\{y_{j_i}\}) = \cup_i A_{j_i},$$

thus $\sigma(Y) \subset \sigma\left(A_1, A_2, \ldots\right)$. Summarizing, we obtained that

$$\sigma(Y) = \sigma\left(A_1, A_2, \ldots\right).$$

1.6. Toss a fair coin 3 times. Let X denote the number of heads, and let $I_i = 1$ if the ith toss is head, 0 otherwise, $i = 1, 2, 3$. Determine the σ-algebras $\sigma(X)$, $\sigma(I_i)$, $i = 1, 2, 3$, $\sigma(I_1, I_2)$, $\sigma(I_1, I_2, I_3)$.

2 Conditional expectation

2.1. Let $(\Omega, \mathcal{A}, \mathbf{P}) = ([0,1], \mathcal{B}_{[0,1]}, \lambda)$, with λ being the Lebesgue measure. Let $\mathcal{F} = \{\emptyset, [0, 1/2), [1/2, 1], [0, 1]\}$. Calculate $\mathbf{E}[X|\mathcal{F}]$.

Solution. Since $\mathbf{E}[X|\mathcal{F}]$ is $\mathcal{F}$-measurable, therefore $= aI_{[0,1/2)}(\omega) + bI_{[1/2,1]}(\omega)$. We have to determine a, b. By definition

$$\int_F X \mathrm{d}\mathbf{P} = \int_F \mathbf{E}(X|\mathcal{F}) \mathrm{d}\mathbf{P},$$

for all $F \in \mathcal{F}$. Clearly, it is enough to check for the sets $[0, /1/2)$ and $[1/2, 1]$. So

$$\begin{aligned}
\int_{[0,1/2)} X \mathrm{d}\mathbf{P} &= \int_{[0,1/2)} \left(aI_{[0,1/2)}(\omega) + bI_{[1/2,1]}(\omega)\right) \mathrm{d}\mathbf{P}(\omega) = \frac{a}{2}, \\
\int_{[1/2,1]} X \mathrm{d}\mathbf{P} &= \int_{[1/2,1]} \left(aI_{[0,1/2)}(\omega) + bI_{[1/2,1]}(\omega)\right) \mathrm{d}\mathbf{P}(\omega) = \frac{b}{2}.
\end{aligned}$$

Therefore, we obtain

$$a = 2 \int_{[0,1/2)} X \mathrm{d}\mathbf{P}, \quad b = 2 \int_{[1/2,1]} X \mathrm{d}\mathbf{P}.$$

2.2. Let $B_1, B_2, \ldots$ be a countable partition of Ω, i.e. they are disjoint, and the union is Ω. Let $\mathcal{G} = \sigma(B_1, B_2, \ldots)$. Determine $\mathbf{E}[X|\mathcal{G}]$.

2.3. Let $(\Omega, \mathcal{A}, \mathbf{P}) = ([-1,1], \mathcal{B}_{[-1,1]}, \lambda/2)$, where λ is the Lebesgue measure. Consider the random variable $X(x) = x^2$. What σ-algebra does X generate? Determine the conditional probability $\mathbf{P}(A|X)$ and conditional expectation $\mathbf{E}[Y|X]$, where A is an event and Y is an integrable random variable. Solve the problem also for the variables $X_2(x) = |x|$ and $X_3(x) = x^6$! (Is there any difference?)

Solution. The function x^2 is even, meaning it does not distinguish between x and $-x$. From this, we can intuitively expect the result: the $\sigma(X)$ consists of sets A that can be written in the form $B \cup (-B)$, where B is any Borel set in $[0,1]$, and $-B = \{-x : x \in B\}$. Formally,

$$\sigma(X) = \{B \cup (-B) \,:\, B \in \mathcal{B}([0,1])\}\,.$$

Indeed, on one hand, if C is an arbitrary Borel set in $\mathbb{R}$, then

$$X^{-1}(C) = X^{-1}(C \cap [0,1]) = \sqrt{(C \cap [0,1])} \cup \left(-\sqrt{(C \cap [0,1])}\right),$$

and $\sqrt{(C \cap [0,1])} \in \mathcal{B}([0,1])$, since the function $X(x) = x^2$ is measurable, $\sqrt{A} = \{\sqrt{a} : a \in A\}$. On the other hand, for any $B \in \mathcal{B}([0,1])$,

$$X^{-1}(B^2) = B \cup (-B),$$

proving the equality.

Now, we determine the conditional expectations with respect to this σ-algebra. By definition,

$$\mathbf{P}(A|X) = \mathbf{E}[I_A|X] = \mathbf{E}[I_A|\sigma(X)].$$

The information from $\sigma(X)$ tells us that for each $x \in [-1,1]$, we can determine whether $\{x, -x\}$ has occurred or not. Based on this: (i) If x is such that $x \notin A$ and $-x \notin A$, then we are sure that A has not occurred, so for such x, we have $\mathbf{P}(A|X)(x) = 0$. (ii) If $x \in A$ and $-x \in A$, then we are certain that A has occurred, thus $\mathbf{P}(A|X)(x) = 1$. (iii) Finally, if $x \in A$ but $-x \notin A$, or vice versa, then we only know that one of $\{-x, x\}$ has occurred. In one case, A has occurred; in the other, it has not. Summarizing, the natural guess

$$\mathbf{P}(A|X)(x) = \frac{1}{2}.$$

Formally,

$$\mathbf{P}(A|X)(x) = \frac{I_A(x) + I_{-A}(x)}{2}.$$

It is easy to see that the variable on the right-hand side is measurable with respect to $\sigma(X)$, since it takes values in $\{0, 1/2, 1\}$, and the corresponding inverse sets are symmetric about the origin.

Let $B \in \sigma(X)$. We need to show that

$$\int_B I_A(x)\mathrm{d}\mathbf{P}(x) = \int_B \frac{I_A(x) + I_{-A}(x)}{2}\mathrm{d}\mathbf{P}(x).$$

The left-hand side equals $\mathbf{P}(A \cap B)$, while the right-hand side equals

$$\frac{\mathbf{P}(A \cap B) + \mathbf{P}(-A \cap B)}{2},$$

which, due to the symmetry of B, equals $\mathbf{P}(A \cap B)$.

Similarly,

$$\mathbf{E}[Y|X] = \frac{Y + \widetilde{Y}}{2},$$

where

$$\widetilde{Y}(\omega) = Y(-\omega).$$

The measurability and integral properties can be justified in the same way as in the case of $Y = I_A$.

For the random variables X_2 and X_3, there is no difference, since

$$\sigma(X) = \sigma(X_2) = \sigma(X_3).$$

2.4. Let (X, Y) be a jointly continous random vector with density h. Put f_X, f_Y for the marginal densities, and $f_{X|Y}(x|y)$ for the conditional density. Show that

$$\mathbf{E}[X|Y] = \int_{\mathbb{R}} x f_{X|Y}(x|Y)\mathrm{d}x.$$

2.5. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and X, Y random variables. Show that $X \leq Y$ a.s. if and only if for any $A \in \mathcal{A}$

$$\int_A X\mathrm{d}\mathbf{P} \leq \int_A Y\mathrm{d}\mathbf{P}.$$

2.6. Let X, Y be independent, identically distributed. Determine $\mathbf{E}[X|X+Y]$. (First guess, then prove!)

Solution. The natural guess is

$$\mathbf{E}[X|X+Y] = \frac{X+Y}{2}.$$

This is $\sigma(X+Y)$-measurable. We have to show that for each $F \in \sigma(X+Y)$

$$\int_F X \mathrm{d}\mathbf{P} = \int_F \frac{X+Y}{2} \mathrm{d}\mathbf{P}.$$

It is enough to check for sets of the form $F = \{X + Y \leq z\}$, since this is a generating π-system. We have to prove

$$\int_{\{X+Y\leq z\}} X \mathrm{d}\mathbf{P} = \int_{\{X+Y\leq z\}} Y \mathrm{d}\mathbf{P}. \tag{F}$$

Using the transformation theorem, and that X and Y are iid

$$\int_{\{X+Y\leq z\}} X \mathrm{d}\mathbf{P} = \iint_{S_z} x \,\mathrm{d}F(x)\,\mathrm{d}F(y) = \int_{-\infty}^{\infty} xF(z-x)\mathrm{d}F(x),$$

where $S_z = \{(x, y) : x + y \leq z\}$. The latter equals the RHS of (F).

2.7. Let X, Y be iid random variables, with a continuous distribution function F. Put $M = \max\{X, Y\}$. Determine $\mathbf{P}(X \leq x|M)$. (Guess and prove.)

Solution. We know the maximum $M = m$. If $x \geq m$, then $\mathbf{P}(X \leq x|M = m) = 1$, since $X \leq M = m$. If $x < m$, then M is eather X or Y with equal probability. If $X = M$, then $\{X \leq x\}$ does not occur. While, if $Y = M$, then we already know that $X \leq m$, so $\mathbf{P}(X \leq x|X \leq m) = F(x)/F(m)$. Summarizing,

$$\mathbf{P}(X \leq x|M)(\omega) = I_{\{x\geq M(\omega)\}}(\omega) + I_{\{x<M\}}(\omega)\frac{1}{2}\frac{F(x)}{F(M(\omega))}.$$

We show that this indeed holds.

Clearly, the RHS, a function of M, is $\sigma(M)$-measurable. We need to prove that for all $A \in \sigma(M)$

$$\mathbf{P}\left(A \cap \{X \leq x\}\right) = \int_A \left(I_{\{x\geq M(\omega)\}}(\omega) + I_{\{x<M\}}(\omega)\frac{1}{2}\frac{F(x)}{F(M(\omega))} \right) \mathrm{d}\mathbf{P}(\omega).$$

Furthermore, it is enough to check on a generating π-system, so we may assume that $A = \{M \leq m\}$. On the one hand

$$\int_{\{M \leq m\}} I_{\{x \geq M(\omega)\}}(\omega) \mathrm{d}\mathbf{P}(\omega) = \mathbf{P}(M \leq m \wedge x).$$

For $m < x$ the second term is $= 0$, otherwise

$$\begin{aligned}
\int_{\{M \leq m\}} \frac{I_{\{x<M\}}(\omega)}{F(M(\omega))} \mathrm{d}\mathbf{P}(\omega) &= \iint \frac{I_{\{u \vee v \leq m\}}(u,v) I_{\{x < u \vee v\}}(u,v)}{F(u \vee v)} \mathrm{d}F(u) \mathrm{d}F(v) \\
&= 2 \int_x^m \left[\int_{-\infty}^v \frac{1}{F(v)} \mathrm{d}F(u) \right] \mathrm{d}F(v) \\
&= 2(F(m) - F(x)).
\end{aligned}$$

Thus the RHS

$$= \begin{cases} \mathbf{P}(M \leq m) = F(m)^2, & \text{if } m < x, \\ \mathbf{P}(M \leq x) + F(x)(F(m) - F(x)) = F(m)F(x), & \text{if } m \geq x. \end{cases}$$

The LHS

$$\mathbf{P}(\{M \leq m\} \cap \{X \leq x\}) = \begin{cases} \mathbf{P}(M \leq m) = F(m)^2, & \text{if } m < x, \\ \mathbf{P}(Y \leq m, X \leq x) = F(m)F(x), & \text{if } m \geq x, \end{cases}$$

proving the statement.

2.8. Let N be a Poisson random variable with parameter λ. Throw a dice N times, and let X denote the number of 6-es. Determine the distribution of X. Determine the conditional distribution of X given $N = n$.

2.9. Let X, Y iid nonnegative random variables, with continuous distribution functoin F, and let $Z := \max\{X, Y\}$. Determine $\mathbf{P}(X + Y \leq x | Z)$.

2.10. Let X be $\mathcal{G}$-measurable, Y independent of $\mathcal{G}$, $h : \mathbb{R}^2 \to \mathbb{R}$ Borel-measurable. Show that

$$\mathbf{E}\left[h(X,Y)|\mathcal{G}\right] = \int h(X,y) \mathrm{d}G(y),$$

where $G(y) = \mathbf{P}(Y \leq y)$. In particular, if X and Y are independent, then

$$\mathbf{E}[h(X,Y)|X = x] = \mathbf{E}h(x,Y).$$

(Prove first for function of the form $h(x,y) = I_A(x) \cdot I_B(y)$.)

2.11. Let X, Y be random variables on $(\Omega, \mathcal{A}, \mathbf{P})$, and let $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-algebra such that X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$-measurable. Show that for any Borel set A a.s.

$$\mathbf{P}(X + Y \in A|\mathcal{G}) = \mathbf{P}(X + Y \in A|Y),$$

and

$$\mathbf{P}(X + Y \in A|Y = y) = \mathbf{P}(X + y \in A), \quad \mathbf{P}Y^{-1} - \text{a.s.}$$

2.12. Let (U, X) and Y be independent, where U, X, Y are integrable random variables. Show that $\mathbf{E}[U|X, Y] = \mathbf{E}[U|X]$.

Solution. As $\mathbf{E}[U|X]$ is $\sigma(X, Y)$-measurable, we have to show that for any $F \in \sigma(X, Y)$

$$\int_F \mathbf{E}[U|X] \mathrm{d}\mathbf{P} = \int_F U \mathrm{d}\mathbf{P}.$$

We may assume that $F = I(X \in A, Y \in B)$. Then

$$\begin{aligned}
&\int_F \mathbf{E}[U|X] \mathrm{d}\mathbf{P} = \int_\Omega I(X \in A) \mathbf{E}[U|X] \, I(Y \in B) \mathrm{d}\mathbf{P} \\
&= \int_\Omega I(X \in A) \mathbf{E}[U|X] \mathrm{d}\mathbf{P} \int_\Omega I(Y \in B) \mathrm{d}\mathbf{P} \qquad \text{(indpendence)} \\
&= \int_\Omega I(X \in A) U \mathrm{d}\mathbf{P} \int_\Omega I(Y \in B) \mathrm{d}\mathbf{P} \qquad \text{(conditional expectation)} \\
&= \int_\Omega I(X \in A) U \, I(Y \in B) \mathrm{d}\mathbf{P} \qquad \text{(independence)} \\
&= \int_F U \mathrm{d}\mathbf{P}.
\end{aligned}$$

2.13. Let μ and ν be finite measure on $(\Omega, \mathcal{A})$ such that $\mu(\Omega) = \nu(\Omega)$. Show that

$$\mathcal{L} = \{A : \mu(A) = \nu(A)\}$$

is λ-system.

2.14. Let Z be a standard normal random variable, and $t \in \mathbb{R}$. Calculate $\mathbf{E}[Z| \min\{Z, t\}]$.

2.15. Define the conditional variance of X as

$$\mathbf{Var}(X|\mathcal{G}) = \mathbf{E}[X^2|\mathcal{G}] - (\mathbf{E}[X|\mathcal{G}])^2.$$

Prove that

$$\mathbf{Var}(X) = \mathbf{E}\mathbf{Var}(X|\mathcal{G}) + \mathbf{Var}(\mathbf{E}[X|\mathcal{G}]).$$

2.16. Let $(\Omega, \mathcal{A}, \mathbf{P}) = ([0,1], \mathcal{B}_{[0,1]}, \lambda)$, where λ is the Lebesgue measure, and let $\mathcal{G}$ the collection of those sets which are countable or co-countable, i.e. $\mathcal{G} = \{A \subset [0,1] \, : \, A \text{ or } A^c \text{ is countable}\}$. Show that $\mathbf{P}(A|\mathcal{G}) = \mathbf{P}(A)$! (This goes agains the intiution, as $\{x\} \in \mathcal{G}$ for each x, that is we can decide on each single x, whether it happened or not.)

2.17. Let g be the density of Y, f the density of X. Prove that

$$g(y) = \int_{-\infty}^{\infty} g(y|x) f(x) \, \mathrm{d}x.$$

2.18. Let (X, Y) be a uniform random variable in the unit disc. Determine the conditional distribution of Y given $X = x$! Calculate $\mathbf{E}[Y^2 | X = x]$!

2.19. Let X, Y, Z be indendent exponentials with parameters λ, μ, and ν. Calculate the probabilities $\mathbf{P}(X > Y)$, $\mathbf{P}(X > Y > Z)$.

2.20. Let X, Y be independent random variables following an exponential distribution with parameter 1. Let $S = X + Y$ denote their sum. Determine the conditional distribution of X given S.

Provide the conditional density function and identify the distribution! Conversely, determine the conditional distribution of S given X, provide its density function, and identify the distribution.

Compute the conditional expectations

$$\mathbf{E}[X^k | S = s], \quad \mathbf{E}[S^k | X = x], \quad k = 1, 2.$$

2.21. Let $X, X_1, \ldots, X_{n+1}$ iid Exp(1) random variables. Let $S_k = X_1 + \ldots + X_k$. Determine the distribution of $(S_1, \ldots, S_n)$ conditioned on S_{n+1}.

As a consequence, show that

$$\left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}} \right) \stackrel{\mathcal{D}}{=} (U_{1,n}, \ldots, U_{n,n}) \, ,$$

where $0 \leq U_{1,n} \leq \ldots \leq U_{n,n}$ is the order statistic of n iid Uniform$(0,1)$ random variables.

Solution. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be the matrix for which $A(X_1, \ldots, X_{n+1})^\top = (S_1, \ldots, S_{n+1})^\top$. Clearly, A is a lower-diagonal matrix with all nonzero elements equal to 1. Then

$$\begin{aligned}\mathbf{P}((S_1,\ldots,S_{n+1})^\top \in F) &= \mathbf{P}((X_1,\ldots,X_{n+1})^\top \in A^{-1}F)\\ &= \int_{A^{-1}F} e^{-(y_1+\ldots+y_{n+1})}\mathrm{d}\mathbf{y}\\ &= \int_F e^{-u_{n+1}} I(0\le u_1\le\ldots\le u_{n+1})\mathrm{d}\mathbf{u}.\end{aligned}$$

Therefore, the joint density of $(S_1,\ldots,S_{n+1})$ is

$$h_{n+1}(u_1,\ldots,u_{n+1}) = e^{-u_{n+1}} I(0\le u_1\le\ldots\le u_{n+1}).$$

Thus the marginal density of S_{n+1}

$$f_{n+1}(u) = \int_{\mathbb{R}^n} h_{n+1}(u_1,\ldots,u_n,u)\mathrm{d}\mathbf{u} = \frac{u^n}{n!}e^{-u}.$$

(We already knew, that it is a Gamma$(n,1)$.) Thus the conditional density

$$\begin{aligned}&h_{(S_1,\ldots,S_n)|S_{n+1}}(u_1,\ldots,u_n|u_{n+1})\\ &= \frac{h_{n+1}(u_1,\ldots,u_{n+1})}{f_{n+1}(u_{n+1})} = I(u_1\le\ldots\le u_n)\frac{n!}{u_{n+1}^n}.\end{aligned}$$

This is the uniform distribution on the simplex $(u_1\le\ldots\le u_n\le u_{n+1})$.

2.22. Let X,Y be independent exponential random variables with parameter 1. Let $M=\max\{X,Y\}$ and $S=X+Y$. Determine the conditional density function of S given M, and of M given S.

Solution. Clearly, $S\ge M\ge S/2\ge 0$. Let m,s such that $0<s/2<m<s$. Introduce the notation

$$T_{s,m} = \{(u,v):\ 0\le u,v\le m,\ u+v\le s\}.$$

By symmetry

$$\begin{aligned}\mathbf{P}(S\le s, M\le m) &= \iint_{T_{s,m}} e^{-u}e^{-v}\mathrm{d}u\mathrm{d}v\\ &= 2\left[\int_0^{s/2}\int_0^u e^{-u-v}\mathrm{d}v\mathrm{d}u + \int_{s/2}^m\int_0^{s-u} e^{-u-v}\mathrm{d}v\mathrm{d}u\right]\\ &= 1-2e^{-m}+e^{-s}+(2m-s)e^{-s}.\end{aligned}$$

Differentiating, we obtain that the joint density

$$f_{S,M}(s,m) = \begin{cases} 2e^{-s}, & \text{if } 0 < s/2 < m < s, \\ 0, & \text{otherwise.} \end{cases}$$

From here we obtain the marginals, as

$$f_S(s) = \int_{-\infty}^{\infty} f_{S,M}(s,m)\mathrm{d}m = \int_{s/2}^{s} 2e^{-s}\mathrm{d}m = se^{-s},\ s > 0,$$

$$f_M(m) = \int_{-\infty}^{\infty} f_{S,M}(s,m)\mathrm{d}s = \int_{m}^{2m} 2e^{-s}\mathrm{d}s = 2\left(e^{-m} - e^{-2m}\right).$$

The conditional densities

$$g_{S|M}(s|m) = \frac{f_{S,M}(s,m)}{f_M(m)} = \frac{e^{-s}}{e^{-m} - e^{-2m}}, \quad \text{if } m \le s \le 2m,$$

$$g_{M|S}(m|s) = \frac{f_{S,M}(s,m)}{f_S(s)} = \frac{2}{s}, \quad \text{if } s/2 \le m \le s.$$

That is, given the sum S, the maximum is uniform on $(S/2, S)$.

We can also determine the conditional expectations

$$\mathbf{E}[S|M = m] = \int_{-\infty}^{\infty} s g_{S|M}(s|m)\mathrm{d}s = m + 1 - \frac{m}{e^m - 1}.$$

2.23. Let X, Y be iid standard exponentials (i.e. with parameter 1). Put $U = X \wedge Y$, $V = X \vee Y$. Determine and recognise the conditional densities $g_{U|V}(u|v)$, $g_{V|U}(v|u)$.

2.24. Let X, Y be iid random variables with density f. Put $U = X \wedge Y$, $V = X \vee Y$. Determine the conditional densities $g_{U|V}(u|v)$, $g_{V|U}(v|u)$.

Solution. As $U \le V$, $f_{U,V}(u,v)$ is 0, for $v < u$. If $v \ge u$, then

$$\mathbf{P}(U \le u, V \le v) = F(u)(2F(v) - F(u)),$$

where F is the common distribution function. Thus

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v}\mathbf{P}(U \le u, V \le v) = 2f(u)f(v),\ u \le v.$$

The marginals

$$f_U(u) = 2[1 - F(u)]f(u),$$
$$f_V(v) = 2F(v)f(v).$$

Thus the conditional densities

$$g_{V|U}(v|u) = \frac{f_{U,V}(u,v)}{f_U(u)} = \frac{f(v)}{1 - F(u)}, \; v \geq u,$$
$$g_{U|V}(u|v) = \frac{f_{U,V}(u,v)}{f_V(v)} = \frac{f(u)}{F(v)}, \; u \leq v.$$

2.25 (Chebyshev's inequality)**.** Prove that

$$\mathbf{P}(|X| \geq x|\mathcal{G}) \leq \frac{\mathbf{E}[X^2|\mathcal{G}]}{x^2}.$$

2.26. From the uniform distribution on $(0,1)$ choose a random value p. Consider a biased coin, where the probability of heads is p. Let us toss this coin until the first heads, and let X denote the number of tosses. Determine the distribution of X and $\mathbf{E}(X)$.

2.27. Let X be uniform on $(0,1)$ and given $X = x$ let Y be uniform on $(0,x)$. Determine the joint distribution (X,Y), the marginals, expectation, and the covariance matrix!

2.28. Let λ be a uniform on $(0,1)$, and given $\lambda = \lambda_0$ let X be $\mathrm{Exp}(\lambda_0)$. Calculate the distribution function of X.

2.29. Let $X, X_1, X_2, \ldots$ be iid nonnegative random variables with finite mean. Fix $A > 0$ such that $\alpha = \mathbf{P}(X > A) > 0$, and put $N = \min\{k : X_k > A\}$.

(a) Determine the distribution of N.

(b) Show that $\mathbf{E}X_N = \alpha^{-1} \int_{(A,\infty)} x \mathrm{d}F(x)$.

(c) Show that if $X \sim \mathrm{Exp}(\lambda)$ then $\mathbf{E}X_N = A + \lambda^{-1}$.

(d) Prove that N and X_N are independent.

2.30. Let N be Poisson(λ) and $p \in (0,1)$. Given N let $X \sim \mathrm{Binom}(N,p)$. Show that X and $N - X$ are independent, and X has Poisson distribution with parameter $p\lambda$.

2.31. Let N be a nonnegative integer-valued random variable and $p \in (0,1)$. Given N let $X \sim \text{Binom}(N,p)$. Assume that X and $N-X$ are independent. Show that N has Poisson distribution.

Solution. Put $Y = N - X$. Given $N = n$

$$\mathbf{E}[s^X t^Y | N = n] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k t^{n-k} = (sp + t(1-p))^N.$$

Let $f(s) = \mathbf{E}s^N$. Then

$$\mathbf{E}(s^X t^Y) = \mathbf{E}((sp + t(1-p))^N) = f(sp + t(1-p)).$$

Since X and Y are independent

$$\mathbf{E}(s^X t^Y) = \mathbf{E}(s^X)\mathbf{E}(t^Y) = f(sp + 1 - p) f(p + t(1-p)).$$

Thus we obtain the functional equation

$$f(sp + t(1-p)) = f(sp + 1 - p) f(p + t(1-p)).$$

Putting $g(s) = \log f(s)$,

$$g(sp + t(1-p)) = g(sp + 1 - p) + g(p + t(1-p)).$$

Differentiate with respect to s

$$g'(sp + t(1-p)) = g'(sp + 1 - p).$$

Since RHS does not depend on t we see that g has to be linear, that is for some $a, b \in \mathbb{R}$

$$g(s) = as + b.$$

Thus $f(s) = e^{as+b}$. Then $f(1) = 1$ implies that $a + b = 0$, and the monotinicity of f implies that $a = \lambda > 0$. Therefore $f(s) = e^{\lambda(s-1)}$, which is the generating function of the Poisson distribution.

2.32. Let N be Poisson(λ) and $A \in [0,1]$, such that N and A are indpendent. Given (N, A) let $X \sim \text{Binom}(N, A)$. Let $Y = N - X$. Show that $\mathbf{Cov}(X,Y) = -\lambda^2 \mathbf{Var}(A)$.

(Combined with exercise 2.30 this means that X and Y are independent iff A is deterministic.)

Solution. By the tower rule

$$\mathbf{E}X = \mathbf{E}\left(\mathbf{E}[X|A]\right) = \mathbf{E}(\lambda A) = \lambda \mathbf{E}A.$$

Using Exercise 2.30 $\mathcal{L}(X|A) \sim \text{Poisson}(\lambda A)$. By the conditional variance formula (Exercise 2.15)

$$\mathbf{Var}(X) = \mathbf{E}(\mathbf{Var}(X|A)) + \mathbf{Var}(E[X|A]) = \lambda \mathbf{E}(A) + \lambda^2 \mathbf{Var}(A),$$

and similarly $\mathcal{L}(Y|A) \sim \text{Poisson}(\lambda(1-A))$, and

$$\mathbf{Var}(Y) = \mathbf{E}(\mathbf{Var}(Y|A)) + \mathbf{Var}(E[Y|A]) = \lambda(1 - \mathbf{E}(A)) + \lambda^2 \mathbf{Var}(A).$$

Finally,

$$\mathbf{Var}(N) = \mathbf{Cov}(X+Y, X+Y) = \mathbf{Var}(X) + \mathbf{Var}(Y) + 2\mathbf{Cov}(X,Y),$$

thus

$$\mathbf{Cov}(X,Y) = \frac{1}{2}\left(\mathbf{Var}(N) - \mathbf{Var}(X) - \mathbf{Var}(Y)\right) = -\lambda^2 \mathbf{Var}(A).$$

3 Martingales

3.1. Let $\xi, \xi_1, \xi_2, \ldots$ independent, identically distributed random variables with $\mathbf{E}\xi = 0$. Show that $X_n = \xi_1 + \cdots + \xi_n$ is martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(\xi_i : i = 1, 2, \ldots, n)$.

Solution. Clearly X_n is $\mathcal{F}_n$ measurable. By the assumption $\mathbf{E}|\xi| < \infty$, thus $\mathbf{E}|X_n| \leq n\mathbf{E}|\xi| < \infty$. Finally, as ξ_{n+1} is independent of $\xi_1, \ldots, \xi_n$,

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = \mathbf{E}[X_n + \xi_{n+1}|\mathcal{F}_n] = X_n + \mathbf{E}[\xi_{n+1}|\mathcal{F}_n] = S_n.$$

3.2. Let $\xi, \xi_1, \xi_2, \ldots$ iid random variables, such that $\mathbf{P}(\xi = 2) = \frac{1}{3}$, $\mathbf{P}(\xi = \frac{1}{2}) = \frac{2}{3}$. Let $\tau = \min\{k > 0 : X_k = 2\}$. Show that $X_n = \xi_1 \xi_2 \ldots \xi_n$ is a martingale and τ is a stopping time.

3.3. Let $Y, Y_1, Y_2, \ldots$ iid nonnegative random variables with $\mathbf{E}Y = 1$. Show that $X_n = \prod_{i=1}^n Y_i$ is a martingale, with respect to the filtration $\mathcal{F}_n = \sigma(Y_i : i = 1, 2, \ldots, n)$.

3.4. Let $Y, Y_1, Y_2, \ldots$ be iid random variables, such that $\mathbf{P}(Y = 1/2) = \mathbf{P}(Y = 3/2) = 1/2$. Then $X_n = \prod_{i=1}^n Y_i$ is a martingale. Show that $X_n \to 0$ a.s.

This is an example that $\mathbf{E}X_\infty \neq \lim_{n\to\infty} \mathbf{E}X_n$.

3.5. Pólya urn. An urn contains a red and a green ball. Each time we draw a ball randomly with replacement, and additionally add 1 ball of the same color. Let X_n denote the fraction of green balls after the nth draw. Show that X_n is a martingale, and determine the distributions of $\lim X_n$.

3.6. Let $\mathcal{F}_n$ be an arbitrary filtration in $\mathcal{A}$, and X an integrable random variable. Show that $X_n = \mathbf{E}[X|\mathcal{F}_n]$, $n = 1, 2, \ldots$, is a martingale with respect to the filtration $\mathcal{F}_n$.

Solution. Measurability and integrability are clear. Using the tower rule

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = \mathbf{E}[\mathbf{E}[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathbf{E}[X|\mathcal{F}_n] = X_n.$$

3.7. Let $\mathcal{F}_n$ be *any* sequence of σ-algebras and let X be an integrable random variable. Show that the sequence

$$X_n = \mathbf{E}[X|\mathcal{F}_n],$$

is uniformly integrable.

3.8. Let (X_n) be a submartingale, such that $\mathbf{E}X_n = \mathbf{E}X_0$. Show that (X_n) is a martingale!

3.9. Let $X, X_1, X_2, \ldots$ iid random variables with $\mathbf{E}X \geq 0$. Show that $S_n = X_1 + \cdots + X_n$ is a submartingale, and determine its Doob decomposition.

3.10. Let $X, X_1, X_2, \ldots$ be iid random variables with $\mathbf{E}X = 0$, and $\mathbf{E}X^2 = \sigma^2 < \infty$. Show that $S_n^2 = (X_1 + \cdots + X_n)^2$ is a submartingale, and determine its Doob decomposition.

3.11. Let (X_n) be a square integrable martingale and (H_n) a predictable sequence. Let (A_n) be the increasing process of X. Determine the increasing process of $(H \cdot X)$ (in terms of H and A).

3.12 (Records.)**.** Let $\xi_1, \xi_2, \ldots$ be iid continuous random variables. A record occurs at time n if $\xi_n > \max\{\xi_1, \ldots, \xi_{n-1}\}$. Let R_n denote the number of records until time n. Show that

$$\lim_{n\to\infty} \frac{R_n}{\log n} = 1 \quad \text{a.s.}$$

(Use the strong second BC lemma.)

3.13. Let $X, X_1, X_2, \ldots$ iid random signs with $\mathbf{P}\{X = \pm 1\} = 1/2$. Then $S_n = X_1 + \cdots + X_n$ is the simple symmetric random walk. Show that $\tau = \min_{n \in \mathbb{N}} \{S_n = 1\}$ is a stopping time. By Pólya's theorem $\mathbf{P}(\tau < \infty) = 1$. Show that $\mathbf{E}\tau = \infty$.
Hint. Use Wald's identity, and realize that you cannot use it!

3.14. Gambler's ruin $p = \frac{1}{2}$. Let $X, X_1, X_2, \ldots$ be iid random variables such that $\mathbf{P}(X = \pm 1) = \frac{1}{2}$, and put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Fix $a, b \in \mathbb{N}$ and let

$$\tau = \tau_{a,b} = \inf\{n : S_n \geq b \text{ or } S_n \leq -a\}.$$

Show that τ is a stopping time, and $\tau < \infty$ a.s. Determine the probabilities $\mathbf{P}(S_\tau = b)$, $\mathbf{P}(S_\tau = -a)$. Calculate $\mathbf{E}\tau$.

3.15. Let $X, X_1, X_2, \ldots$ be iid random variables such that $\mathbf{P}(X = 1) = p = 1 - \mathbf{P}(X = -1)$, $p \neq \frac{1}{2}$, and put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Fix $a, b \in \mathbb{N}$ and let

$$\tau = \tau_{a,b}(p) = \inf\{n : S_n \geq b \text{ or } S_n \leq -a\},$$

Show that

$$Z_n = s^{S_n} = \prod_{k=1}^{n} s^{X_k}$$

is a martingale, where $s = (1-p)/p = 1/r$. Determine the probabilities $\mathbf{P}(S_\tau = b)$, $\mathbf{P}(S_\tau = -a)$. Calculate $\mathbf{E}\tau$.

3.16. Let $X, X_1, \ldots$ be iid random variables such that $\mathbf{P}(X = \pm 1) = \frac{1}{2}$, and let S_n denote its partial sum. Let $\tau = \min\{k : |S_k| \geq a\}$, $a \in \mathbb{N}$. Find constants such that $M_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale! Compute $\mathbf{E}\tau^2$.

3.17. Let $X, X_1, \ldots$ be iid random variables such that $\mathbf{P}(X = 1) = p = 1 - \mathbf{P}(X = -1)$, $p \geq \frac{1}{2}$, and let S_n denote its partial sum. Show that $M_n = e^{\theta S_n - n\psi(\theta)}$ is a martingale, where $\psi(\theta) = \log E e^{\theta X} = \log \varphi(\theta)$. Let $\tau_1 = \min\{n : S_n = 1\}$. Show that $\mathbf{E}\varphi(\theta)^{-\tau_1} = e^{-\theta}$. As a consequence

$$\mathbf{E}(s^{\tau_1}) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}.$$

3.18. Let S_n be a simple symmetric random walk, and $\tau_1 = \min\{n : S_n = 1\}$. Show that $\mathbf{E}\frac{1}{\tau_1} = \frac{\pi}{2} - 1$.

(Equivalently, in a coin-tossing terminology, if Y denotes the ratio of heads at the first time when the number of heads exceeds the number of tails, then $\mathbf{E}(Y) = \frac{\pi}{4}$. Indeed, $Y = \frac{1}{2} + \frac{1}{2\tau_1}$.)

Solution. Note that

$$\int_0^1 \frac{\mathbf{E}s^{\tau_1}}{s} \mathrm{d}s = \mathbf{E}\left(\int_0^1 s^{\tau_1 - 1}\mathrm{d}s\right) = \mathbf{E}\frac{1}{\tau_1}.$$

By the previous exercise, the generating function of τ_1 is

$$\mathbf{E}(s^{\tau_1}) = \frac{1-\sqrt{1-s^2}}{s},$$

thus, we just have a evaluate an integral. This can be done as

$$\begin{aligned}\mathbf{E}\frac{1}{\tau_1} &= \int_0^1 \frac{1-\sqrt{1-s^2}}{s^2}\mathrm{d}s = \int_0^1 \frac{1}{1+\sqrt{1-s^2}}\mathrm{d}s \\ &= \int_0^{\pi/2} \frac{\cos x}{1+\cos x}\mathrm{d}x = \frac{\pi}{2} - \int_0^{\pi/2} \frac{1}{1+\cos x}\mathrm{d}x \\ &= \frac{\pi}{2} - \int_0^{\pi/2} \frac{1}{2(\cos x/2)^2}\mathrm{d}x = \frac{\pi}{2} - [\tan x]_0^{\pi/4} = \frac{\pi}{2} - 1.\end{aligned}$$

3.19. Let $\xi, \xi_1, \xi_2, \ldots$ iid random variables, such that $\mathbf{P}(\xi = 0) = \mathbf{P}(\xi = 1) = \frac{1}{2}$. Fix the pattern $A = 01001$, and let τ denote the first time when the pattern appears. The aim is to determine $\mathbf{E}\tau$.

Assume that a new gambler arrives before each time n, and bets 1 ducat that $\xi_n = 0$. If he loses, he leaves the game. If he wins he gets 2 ducats, and continues to play. He bets the whole amount on that $\xi_{n+1} = 1$. In each time the gambler either loses all his money and leaves, or doubles his fortune and continues to bet on the next character of the pattern A. If the gambler is lucky and finishes the pattern A he leaves with $2^5 = 32$ ducats. Let X_n denote the total amount of money collected by the casino from all gamblers up to, and including time n. Show that X_n is a martingale. Calculate X_τ. Use the optimal stopping theorem to determine $\mathbf{E}\tau$.

3.20. Let $\{X_n : 0 \le n \le N\}$ be an adapted process. Show that (X_n) is a martingale iff $\mathbf{E}X_\tau = \mathbf{E}X_0$ for all stopping time τ.

3.21. Let τ be a stopping time for $(\mathcal{F}_n)$. Show that $\mathbf{E}[X|\mathcal{F}_n] = \mathbf{E}[X|\mathcal{F}_\tau]$ on $\{\tau = n\}$.

Solution. We have to show that

$$\mathbf{E}[I(\tau = n)X|\mathcal{F}_\tau] = I(\tau = n)\mathbf{E}[X|\mathcal{F}_n].$$

First we show that the RHS is $\mathcal{F}_\tau$-measurable. Indeed, for any $A \in \mathcal{B}(\mathbb{R})$

$$\{I(\tau = n)\mathbf{E}[X|\mathcal{F}_n] \in A\} = \begin{cases} \{\tau = n\} \cap \{\mathbf{E}[X|\mathcal{F}_n] \in A\}, & 0 \notin A, \\ (\{\tau = n\} \cap \{\mathbf{E}[X|\mathcal{F}_n] \in A\}) \cup \{\tau \neq n\}, & 0 \in A. \end{cases}$$

Therefore, for any $m = 1, 2, \ldots$

$$\{I(\tau = n)\mathbf{E}[X|\mathcal{F}_n] \in A\} \cap \{\tau \leq m\} \in \mathcal{F}_m.$$

To check the integral condition, let $B \in \mathcal{F}_\tau$. Since $B \cap \{\tau = n\} \in \mathcal{F}_n$ we have

$$\int_B I(\tau = n)\mathbf{E}[X|\mathcal{F}_n]\mathrm{d}\mathbf{P} = \int_{B\cap\{\tau=n\}} X\mathrm{d}\mathbf{P} = \int_B I(\tau = n)X\mathrm{d}\mathbf{P}.$$